Problem 20

In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\, dt\right].$$

It follows that W is either always zero or nowhere zero on I.

(d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions y_1, ..., y_n. That is, establish Abel's formula

$$W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right]$$

for this case.