Problem 25

(a) Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$.

(b) Show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.

(c) Show that $W(f, g)(t)$ is zero for all t in $-1 < t < 1$.

Solution

The absolute value of t is defined as

$$|t| = \begin{cases} t & \text{if } t > 0 \\ -t & \text{if } t < 0 \end{cases}.$$

So then

$$t^2|t| = \begin{cases} t^3 & \text{if } t > 0 \\ -t^3 & \text{if } t < 0 \end{cases}$$

$$f(t) = \begin{cases} g(t) & \text{if } t > 0 \\ -g(t) & \text{if } t < 0 \end{cases}.$$

$f(t)$ and $g(t)$ are linearly dependent on the intervals, $0 < t < 1$ and $-1 < t < 0$, because they are constant multiples of one another (1 and -1, respectively). They are not constant multiples of one another over $-1 < t < 1$, though, so here they are linearly independent. The Wronskian of f and g is

$$\begin{aligned} W(f, g) &= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \\ &= \begin{vmatrix} t^2|t| & t^3 \\ 2t|t| + t^2 \operatorname{sgn} t & 3t^2 \end{vmatrix} \\ &= t^2|t|(3t^2) - t^3(2t|t| + t^2 \operatorname{sgn} t) \\ &= 3t^4|t| - 2t^4|t| - t^5 \operatorname{sgn} t \\ &= t^4|t| - t^5 \operatorname{sgn} t \\ &= t^4(|t| - t \operatorname{sgn} t) \\ &= t^4(0) \\ &= 0. \end{aligned}$$

Note that the sign (signum) function is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases},$$

so

$$t \operatorname{sgn} t = \begin{cases} t & \text{if } t > 0 \\ -t & \text{if } t < 0 \end{cases} = |t|.$$