Problem 26

Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1v''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0.$$

Solution

Suppose that $y_1 = y_1(t)$ is one solution to the ODE. According to the method of reduction of order, the general solution can be obtained by plugging in $y(t) = y_1(t)v(t)$ to the ODE.

$$[y_1(t)v(t)]''' + p_1(t)[y_1(t)v(t)]'' + p_2(t)[y_1(t)v(t)]' + p_3(t)[y_1(t)v(t)] = 0$$

Evaluate the derivatives.

$$\begin{aligned}&[y_1'(t)v(t) + y_1(t)v'(t)]'' + p_1(t)[y_1'(t)v(t) + y_1(t)v'(t)]' \\ &\qquad + p_2(t)[y_1'(t)v(t) + y_1(t)v'(t)] + p_3(t)[y_1(t)v(t)] = 0\end{aligned}$$

$$\begin{aligned}&[y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t)]' + p_1(t)[y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t)] \\ &\qquad + p_2(t)[y_1'(t)v(t) + y_1(t)v'(t)] + p_3(t)[y_1(t)v(t)] = 0\end{aligned}$$

$$\begin{aligned}&[y_1'''(t)v(t) + 3y_1''(t)v'(t) + 3y_1'(t)v''(t) + y_1(t)v'''(t)] \\ &\qquad + p_1(t)[y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t)] \\ &\qquad + p_2(t)[y_1'(t)v(t) + y_1(t)v'(t)] + p_3(t)[y_1(t)v(t)] = 0\end{aligned}$$

Expand the left side.

$$\begin{aligned}&y_1'''(t)v(t) + 3y_1''(t)v'(t) + 3y_1'(t)v''(t) + y_1(t)v'''(t) \\ &\qquad + p_1(t)y_1''(t)v(t) + 2p_1(t)y_1'(t)v'(t) + p_1(t)y_1(t)v''(t) \\ &\qquad + p_2(t)y_1'(t)v(t) + p_2(t)y_1(t)v'(t) + p_3(t)y_1(t)v(t) = 0\end{aligned}$$

Factor $v(t)$.

$$\begin{aligned}&3y_1''(t)v'(t) + 3y_1'(t)v''(t) + y_1(t)v'''(t) \\ &\qquad + 2p_1(t)y_1'(t)v'(t) + p_1(t)y_1(t)v''(t) + p_2(t)y_1(t)v'(t) \\ &\qquad + \underbrace{[y_1'''(t) + p_1(t)y_1''(t) + p_2(t)y_1'(t) + p_3(t)y_1(t)]}_{=0}v(t) = 0\end{aligned}$$

The quantity in square brackets is zero because y_1 is a solution to the ODE.

$$3y_1''(t)v'(t) + 3y_1'(t)v''(t) + y_1(t)v'''(t) + 2p_1(t)y_1'(t)v'(t) + p_1(t)y_1(t)v''(t) + p_2(t)y_1(t)v'(t) = 0$$

Therefore, factoring v' and v'',

$$y_1v''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0.$$