Problem 14

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(4)} - 4y''' + 4y'' = 0$$

Solution

Start by making the substitution $u = y''$. Then this fourth-order ODE reduces to a second-order ODE.

$$u'' - 4u' + 4u = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = \{2\}$$

One solution to the ODE is then $u = e^{2t}$. By using the method of reduction of order, we can obtain the general solution. Plug in $u(t) = c(t)e^{2t}$ to the ODE.

$$[c(t)e^{2t}]'' - 4[c(t)e^{2t}]' + 4[c(t)e^{2t}] = 0$$

Evaluate the derivatives.

$$[c'(t)e^{2t} + 2c(t)e^{2t}]' - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4[c(t)e^{2t}] = 0$$

$$[c''(t)e^{2t} + 4c'(t)e^{2t} + 4c(t)e^{2t}] - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4[c(t)e^{2t}] = 0$$

Expand the left side.

$$c''(t)e^{2t} + \cancel{4c'(t)e^{2t}} + \cancel{4c(t)e^{2t}} - \cancel{4c'(t)e^{2t}} - \cancel{8c(t)e^{2t}} + \cancel{4c(t)e^{2t}} = 0$$

$$c''(t)e^{t} = 0$$

Divide both sides by e^t.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

Since $u(t) = c(t)e^{2t}$,

$$u(t) = C_1 te^{2t} + C_2 e^{2t}.$$

Replace $u(t)$ with y'' now.

$$y''(t) = C_1 te^{2t} + C_2 e^{2t}$$

Integrate both sides with respect to t.

$$y'(t) = \frac{C_1}{4}(2t - 1)e^{2t} + \frac{C_2}{2}e^{2t} + C_3$$

Integrate both sides with respect to t once more.

$$\begin{aligned} y(t) &= \frac{C_1}{4}(t-1)e^{2t} + \frac{C_2}{4}e^{2t} + C_3 t + C_4 \\ &= \frac{C_1}{4}te^{2t} - \frac{C_1}{4}e^{2t} + \frac{C_2}{4}e^{2t} + C_3 t + C_4 \\ &= \frac{C_1}{4}te^{2t} + \frac{1}{4}(-C_1 + C_2)e^{2t} + C_3 t + C_4 \end{aligned}$$

Therefore, using C_5 for $C_1/4$ and C_6 for $(-C_1 + C_2)/4$, the general solution is

$$y(t) = C_5 te^{2t} + C_6 e^{2t} + C_3 t + C_4.$$