Problem 20

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(4)} - 8y' = 0$$

Solution

Start by making the substitution $u = y'$.

$$u''' - 8u = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt} \quad \rightarrow \quad u''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - 8(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 8 = 0$$

$$(r-2)(r^2 + 2r + 4) = 0$$

Use the zero product theorem.

$$\begin{aligned} r - 2 = 0 \quad &\text{or} \quad r^2 + 2r + 4 = 0 \\ r = 2 \quad &\text{or} \quad r = \frac{-2 \pm \sqrt{4 - 4(4)}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} \\ & \qquad r = -1 \pm i\sqrt{3} \end{aligned}$$

$$r = \{2, -1 - i\sqrt{3}, -1 + i\sqrt{3}\}$$

Three solutions to the ODE are then $y = e^{2t}$ and $y = e^{(-1-i\sqrt{3})t}$ and $y = e^{(-1+i\sqrt{3})t}$. By the principle of superposition, the general solution for u is a linear combination of these three.

$$u(t) = C_1 e^{2t} + C_2 e^{(-1-i\sqrt{3})t} + C_3 e^{(-1+i\sqrt{3})t}$$

Change back to y now.

$$y'(t) = C_1 e^{2t} + C_2 e^{(-1-i\sqrt{3})t} + C_3 e^{(-1+i\sqrt{3})t}$$

Integrate both sides with respect to t.

$$\begin{aligned} y(t) &= \frac{C_1}{2} e^{2t} + \frac{C_2}{-1 - i\sqrt{3}} e^{(-1-i\sqrt{3})t} + \frac{C_3}{-1 + i\sqrt{3}} e^{(-1+i\sqrt{3})t} + C_4 \\ &= \frac{C_1}{2} e^{2t} + \frac{C_2}{-1 - i\sqrt{3}} e^{-t-i\sqrt{3}t} + \frac{C_3}{-1 + i\sqrt{3}} e^{-t+i\sqrt{3}t} + C_4 \\ &= \frac{C_1}{2} e^{2t} + \frac{C_2}{-1 - i\sqrt{3}} e^{-t} e^{-i\sqrt{3}t} + \frac{C_3}{-1 + i\sqrt{3}} e^{-t} e^{i\sqrt{3}t} + C_4 \end{aligned}$$

Write $y(t)$ in terms of real functions by using Euler's formula.

$$\begin{aligned} y(t) &= \frac{C_1}{2}e^{2t} + e^{-t}\left(\frac{C_2}{-1-i\sqrt{3}}e^{-i\sqrt{3}t} + \frac{C_3}{-1+i\sqrt{3}}e^{i\sqrt{3}t}\right) + C_4 \\ &= \frac{C_1}{2}e^{2t} + e^{-t}\left[\frac{C_2}{-1-i\sqrt{3}}(\cos\sqrt{3}t - i\sin\sqrt{3}t) + \frac{C_3}{-1+i\sqrt{3}}(\cos\sqrt{3}t + i\sin\sqrt{3}t)\right] + C_4 \\ &= \frac{C_1}{2}e^{2t} + e^{-t}\left[\left(\frac{C_2}{-1-i\sqrt{3}} + \frac{C_3}{-1+i\sqrt{3}}\right)\cos\sqrt{3}t + \left(-i\frac{C_2}{-1-i\sqrt{3}} + i\frac{C_3}{-1+i\sqrt{3}}\right)\sin\sqrt{3}t\right] + C_4 \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = C_5e^{2t} + e^{-t}(C_6\cos\sqrt{3}t + C_7\sin\sqrt{3}t) + C_4.$$