Problem 25

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$18y''' + 21y'' + 14y' + 4y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$18(r^3 e^{rt}) + 21(r^2 e^{rt}) + 14(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$18r^3 + 21r^2 + 14r + 4 = 0$$

$$(2r + 1)(9r^2 + 6r + 4) = 0$$

Use the zero product theorem.

$$\begin{aligned} 2r + 1 = 0 \quad &\text{or} \quad 9r^2 + 6r + 4 = 0 \\ r = -\frac{1}{2} \quad &\text{or} \quad r = \frac{-6 \pm \sqrt{36 - 4(9)(4)}}{2(9)} = \frac{-6 \pm \sqrt{-108}}{18} = \frac{-6 \pm 6i\sqrt{3}}{18} = -\frac{1}{3} \pm i\frac{\sqrt{3}}{3} \end{aligned}$$

$$r = \left\{ -\frac{1}{2}, -\frac{1}{3} - i\frac{\sqrt{3}}{3}, -\frac{1}{3} + i\frac{\sqrt{3}}{3} \right\}$$

Three solutions to the ODE are then $y = e^{-t/2}$ and $y = e^{(-1/3 - i\sqrt{3}/3)t}$ and $y = e^{(-1/3 + i\sqrt{3}/3)t}$. By the principle of superposition, the general solution for y is a linear combination of these three.

$$\begin{aligned} y(t) &= C_1 e^{-t/2} + C_2 e^{(-1/3 - i\sqrt{3}/3)t} + C_3 e^{(-1/3 + i\sqrt{3}/3)t} \\ &= C_1 e^{-t/2} + C_2 e^{-t/3 - i\sqrt{3}t/3} + C_3 e^{-t/3 + i\sqrt{3}t/3} \\ &= C_1 e^{-t/2} + C_2 e^{-t/3} e^{-i\sqrt{3}t/3} + C_3 e^{-t/3} e^{i\sqrt{3}t/3} \\ &= C_1 e^{-t/2} + e^{-t/3}(C_2 e^{-i\sqrt{3}t/3} + C_3 e^{i\sqrt{3}t/3}) \\ &= C_1 e^{-t/2} + e^{-t/3}\left[C_2\left(\cos\frac{\sqrt{3}}{3}t - i\sin\frac{\sqrt{3}}{3}t\right) + C_3\left(\cos\frac{\sqrt{3}}{3}t + i\sin\frac{\sqrt{3}}{3}t\right)\right] \\ &= C_1 e^{-t/2} + e^{-t/3}\left[(C_2 + C_3)\cos\frac{\sqrt{3}}{3}t + (-iC_2 + iC_3)\sin\frac{\sqrt{3}}{3}t\right] \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = C_1 e^{-t/2} + e^{-t/3}\left(C_4\cos\frac{\sqrt{3}}{3}t + C_5\sin\frac{\sqrt{3}}{3}t\right).$$