Problem 26

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt} \quad \rightarrow \quad y^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - 7(r^3e^{rt}) + 6(r^2e^{rt}) + 30(re^{rt}) - 36(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$$

$$(r+2)(r-3)(r^2 - 6r + 6) = 0$$

Use the zero product theorem.

$$r + 2 = 0 \quad \text{or} \quad r - 3 = 0 \quad \text{or} \quad r^2 - 6r + 6 = 0$$

$$r = -2 \quad \text{or} \quad r = 3 \quad \text{or} \quad r = \frac{6 \pm \sqrt{36 - 4(6)}}{2} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

$$r = \{-2, 3 - \sqrt{3}, 3, 3 + \sqrt{3}\}$$

Four solutions to the ODE are then $y = e^{-2t}$ and $y = e^{(3-\sqrt{3})t}$ and $y = e^{3t}$ and $y = e^{(3+\sqrt{3})t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$y(t) = C_1e^{-2t} + C_2e^{(3-\sqrt{3})t} + C_3e^{3t} + C_4e^{(3+\sqrt{3})t}$$