Problem 28

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2e^{rt} \quad \to \quad y''' = r^3e^{rt} \quad \to \quad y^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 6(r^3e^{rt}) + 17(r^2e^{rt}) + 22(re^{rt}) + 14(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$$

$$(r^2 + 2r + 2)(r^2 + 4r + 7) = 0$$

Use the zero product theorem.

$$\begin{aligned} r^2 + 2r + 2 = 0 \qquad &\text{or} \qquad r^2 + 4r + 7 = 0 \\ r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} \qquad &\text{or} \qquad r = \frac{-4 \pm \sqrt{16 - 4(7)}}{2} \\ r = \frac{-2 \pm \sqrt{-4}}{2} \qquad &\text{or} \qquad r = \frac{-4 \pm \sqrt{-12}}{2} \\ r = \frac{-2 \pm 2i}{2} \qquad &\text{or} \qquad r = \frac{-4 \pm 2i\sqrt{3}}{2} \\ r = -1 \pm i \qquad &\text{or} \qquad r = -2 \pm i\sqrt{3} \end{aligned}$$

$$r = \{-1 - i, -1 + i, -2 - i\sqrt{3}, -2 + i\sqrt{3}\}$$

Four solutions to the ODE are then $y = e^{(-1-i)t}$ and $y = e^{(-1+i)t}$ and $y = e^{(-2-i\sqrt{3})t}$ and $y = e^{(-2+i\sqrt{3})t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$\begin{aligned} y(t) &= C_1e^{(-1-i)t} + C_2e^{(-1+i)t} + C_3e^{(-2-i\sqrt{3})t} + C_4e^{(-2+i\sqrt{3})t} \\ &= C_1e^{-t-it} + C_2e^{-t+it} + C_3e^{-2t-i\sqrt{3}t} + C_4e^{-2t+i\sqrt{3}t} \\ &= C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} + C_3e^{-2t}e^{-i\sqrt{3}t} + C_4e^{-2t}e^{i\sqrt{3}t} \\ &= C_1e^{-t}(\cos t - i\sin t) + C_2e^{-t}(\cos t + i\sin t) + C_3e^{-2t}(\cos\sqrt{3}t - i\sin\sqrt{3}t) + C_4e^{-2t}(\cos\sqrt{3}t + i\sin\sqrt{3}t) \\ &= (C_1 + C_2)e^{-t}\cos t + (-iC_1 + iC_2)e^{-t}\sin t + (C_3 + C_4)e^{-2t}\cos\sqrt{3}t + (-iC_3 + iC_4)e^{-2t}\sin\sqrt{3}t \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = C_5e^{-t}\cos t + C_6e^{-t}\sin t + C_7e^{-2t}\cos\sqrt{3}t + C_8e^{-2t}\sin\sqrt{3}t.$$