Problem 37

Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt} \quad \rightarrow \quad y^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - 1 = 0$$
$$(r+1)(r-1)(r^2+1) = 0$$
$$r = \{-1, 1, -i, i\}$$

Four solutions to the ODE are then $y = e^{-t}$ and $y = e^t$ and $y = e^{-it}$ and $y = e^{it}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$y(t) = C_1 e^{-t} + C_2 e^t + C_3 e^{-it} + C_4 e^{it}$$

Note that hyperbolic sine and hyperbolic cosine are defined as

$$\cosh t = \frac{e^t + e^{-t}}{2}$$
$$\sinh t = \frac{e^t - e^{-t}}{2}.$$

Adding the respective sides of these equations yields

$$\cosh t + \sinh t = e^t.$$

On the other hand, subtracting the respective sides of these equations yields

$$\cosh t - \sinh t = e^{-t}.$$

Substitute these two previous equations and use Euler's formula in the general solution.

$$\begin{aligned} y(t) &= C_1(\cosh t - \sinh t) + C_2(\cosh t + \sinh t) + C_3(\cos t - i \sin t) + C_4(\cos t + i \sin t) \\ &= (C_1 + C_2)\cosh t + (-C_1 + C_2)\sinh t + (C_3 + C_4)\cos t + (-iC_3 + iC_4)\sin t \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Differentiate this solution three times with respect to t.

$$\begin{aligned} y'(t) &= -c_1 \sin t + c_2 \cos t + c_3 \sinh t + c_4 \cosh t \\ y''(t) &= -c_1 \cos t - c_2 \sin t + c_3 \cosh t + c_4 \sinh t \\ y'''(t) &= c_1 \sin t - c_2 \cos t + c_3 \sinh t + c_4 \cosh t \end{aligned}$$

Apply the initial conditions now to determine c_1, c_2, c_3, and c_4.

$$\begin{aligned} y(0) &= c_1 + c_3 = 0 \\ y'(0) &= c_2 + c_4 = 0 \\ y''(0) &= -c_1 + c_3 = 1 \\ y'''(0) &= -c_2 + c_4 = 1 \end{aligned}$$

Solving this system of equations yields $c_1 = -1/2$, $c_2 = -1/2$, $c_3 = 1/2$, and $c_4 = 1/2$. Therefore,

$$\begin{aligned} y(t) &= -\frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{1}{2} \cosh t + \frac{1}{2} \sinh t \\ &= \frac{1}{2}(\cosh t + \sinh t - \cos t - \sin t). \end{aligned}$$