Problem 41

In this problem we indicate one way to show that if $r = r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1 t}$, $te^{r_1 t}$, ..., $t^{s-1}e^{r_1 t}$ are solutions of Eq. (1). This problem extends to nth order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{rt}] = e^{rt}Z(r) \tag{i}$$

and differentiate repeatedly with respect to r, setting $r = r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s, then $Z(r) = (r - r_1)^s q(r)$, where $q(r)$ is a polynomial of degree $n - s$ and $q(r_1) \neq 0$. Show that $Z(r_1)$, $Z'(r_1)$, ..., $Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1) \neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r, show that

$$\begin{gathered}\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}],\\ \vdots \\ \frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}] = L[t^{s-1}e^{rt}]\end{gathered}$$

(c) Show that $e^{r_1 t}$, $te^{r_1 t}$,..., $t^{s-1}e^{r_1 t}$ are solutions of Eq. (1).

Solution

Part (a)

Plug in $r = r_1$ to the formula for $Z(r)$.

$$\begin{aligned}Z(r_1) &= (r_1 - r_1)^s q(r_1)\\ &= 0^s q(r_1)\\ &= 0\end{aligned}$$

Differentiate $Z(r)$ with respect to r to obtain $Z'(r)$.

$$Z'(r) = s(r - r_1)^{s-1}q(r) + (r - r_1)^s q'(r)$$

Plug in $r = r_1$ to the formula.

$$\begin{aligned}Z'(r_1) &= s(r_1 - r_1)^{s-1}q(r_1) + (r_1 - r_1)^s q'(r_1)\\ &= s(0)^{s-1}q(r_1) + (0)^s q'(r_1)\\ &= 0\end{aligned}$$

Differentiate Z'(r) with respect to r to obtain $Z''(r)$.

$$Z''(r) = s(s-1)(r - r_1)^{s-2}q(r) + s(r - r_1)^{s-1}q'(r) + s(r - r_1)^{s-1}q'(r) + (r - r_1)^s q''(r)$$

Plug in $r = r_1$ to the formula.

$$\begin{aligned} Z''(r_1) &= s(s-1)(r_1-r_1)^{s-2}q(r_1) + s(r_1-r_1)^{s-1}q'(r_1) + s(r_1-r_1)^{s-1}q'(r_1) + (r_1-r_1)^s q''(r_1) \\ &= s(s-1)(0)^{s-2}q(r_1) + s(0)^{s-1}q'(r_1) + s(0)^{s-1}q'(r_1) + (0)^s q''(r_1) \\ &= 0 \\ &\vdots \end{aligned}$$

Differentiate $Z(r)$ $s-1$ times with respect to r.

$$Z^{(s-1)}(r) = s(s-1)(s-2)\cdots(3)(2)(r-r_1)^{s-(s-1)}q(r) + \text{terms with factors of } r - r_1$$

Plug in $r = r_1$ to the formula.

$$\begin{aligned} Z^{(s-1)}(r_1) &= s(s-1)(s-2)\cdots(3)(2)(r_1-r_1)^{s-(s-1)}q(r_1) + \text{terms with factors of } r_1 - r_1 \\ &= s(s-1)(s-2)\cdots(3)(2)(0)^1 q(r_1) + \text{terms with factors of } 0 \\ &= 0 \end{aligned}$$

Differentiate $Z^{(s-1)}(r)$ once more with respect to r.

$$\begin{aligned} Z^{(s)}(r) &= s(s-1)(s-2)\cdots(3)(2)(1)(r-r_1)^{s-s}q(r) + \text{terms with factors of } r - r_1 \\ &= s!(r-r_1)^0 q(r) + \text{terms with factors of } r - r_1 \\ &= s!q(r) + \text{terms with factors of } r - r_1 \end{aligned}$$

Plug in $r = r_1$ to the formula.

$$\begin{aligned} Z^{(s)}(r_1) &= s!q(r_1) + \text{terms with factors of } r_1 - r_1 \\ &= s!q(r_1) + \text{terms with factors of } 0 \\ &= s!q(r_1) \\ &\neq 0 \end{aligned}$$

Part (b)

First rewrite the right side of Eq. (i).

$$\begin{aligned} L[e^{rt}] &= e^{rt}Z(r) \\ &= e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n) \\ &= a_0 r^n e^{rt} + a_1 r^{n-1}e^{rt} + \cdots + a_{n-1} r e^{rt} + a_n e^{rt} \\ &= a_0 \frac{d^n}{dt^n}(e^{rt}) + a_1 \frac{d^{n-1}}{dt^{n-1}}(e^{rt}) + \cdots + a_{n-1}\frac{d}{dt}(e^{rt}) + a_n e^{rt} \end{aligned}$$

Differentiate both sides m times with respect to r, where $m = 1, 2, \ldots, s-1$.

$$\begin{aligned} \frac{\partial^m}{\partial r^m} L[e^{rt}] &= \frac{\partial^m}{\partial r^m}\left[a_0 \frac{d^n}{dt^n}(e^{rt}) + a_1 \frac{d^{n-1}}{dt^{n-1}}(e^{rt}) + \cdots + a_{n-1}\frac{d}{dt}(e^{rt}) + a_n e^{rt}\right] \\ &= a_0 \frac{\partial^m}{\partial r^m}\frac{d^n}{dt^n}(e^{rt}) + a_1 \frac{\partial^m}{\partial r^m}\frac{d^{n-1}}{dt^{n-1}}(e^{rt}) + \cdots + a_{n-1}\frac{\partial^m}{\partial r^m}\frac{d}{dt}(e^{rt}) + a_n \frac{\partial^m}{\partial r^m} e^{rt} \end{aligned}$$

Since all partial derivatives of e^{rt} are continuous, the mixed derivatives are equal by Clairaut's theorem.

$$\begin{aligned}
\frac{\partial^m}{\partial r^m}L[e^{rt}] &= a_0\frac{d^n}{dt^n}\left(\frac{\partial^m}{\partial r^m}e^{rt}\right) + a_1\frac{d^{n-1}}{dt^{n-1}}\left(\frac{\partial^m}{\partial r^m}e^{rt}\right) + \cdots + a_{n-1}\frac{d}{dt}\left(\frac{\partial^m}{\partial r^m}e^{rt}\right) + a_n\frac{\partial^m}{\partial r^m}e^{rt} \\
&= a_0\frac{d^n}{dt^n}\left(t^m e^{rt}\right) + a_1\frac{d^{n-1}}{dt^{n-1}}\left(t^m e^{rt}\right) + \cdots + a_{n-1}\frac{d}{dt}\left(t^m e^{rt}\right) + a_n(t^m e^{rt}) \\
&= L[t^m e^{rt}]
\end{aligned}$$

Part (c)

Eq. (1) in the textbook is

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0. \tag{1}$$

Use the results of part (b) together with Eq. (i) to determine $L[e^{rt}]$, $L[te^{rt}]$, $\ldots$, $L[t^{s-1}e^{rt}]$.

$$\begin{aligned}
L[e^{rt}] &= e^{rt}Z(r) \qquad\qquad \text{(i)} \\
L[te^{rt}] &= \frac{\partial}{\partial r}L[e^{rt}] = \frac{\partial}{\partial r}[e^{rt}Z(r)] = te^{rt}Z(r) + e^{rt}Z'(r) \\
L[t^2e^{rt}] &= \frac{\partial^2}{\partial r^2}L[e^{rt}] = \frac{\partial^2}{\partial r^2}[e^{rt}Z(r)] = t^2e^{rt}Z(r) + 2te^{rt}Z'(r) + e^{rt}Z''(r) \\
L[t^3e^{rt}] &= \frac{\partial^3}{\partial r^3}L[e^{rt}] = \frac{\partial^3}{\partial r^3}[e^{rt}Z(r)] = t^3e^{rt}Z(r) + 3t^2e^{rt}Z'(r) + 3te^{rt}Z''(r) + e^{rt}Z'''(r) \\
L[t^4e^{rt}] &= \frac{\partial^4}{\partial r^4}L[e^{rt}] = \frac{\partial^4}{\partial r^4}[e^{rt}Z(r)] = t^4e^{rt}Z(r) + 4t^3e^{rt}Z'(r) + 6t^2e^{rt}Z''(r) + 4te^{rt}Z'''(r) + e^{rt}Z^{(4)}(r) \\
&\vdots \\
L[t^{s-1}e^{rt}] &= \frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}] = \frac{\partial^{s-1}}{\partial r^{s-1}}[e^{rt}Z(r)] = t^{s-1}e^{rt}Z(r) + \cdots + e^{rt}Z^{(s-1)}(r)
\end{aligned}$$

Plugging in $r = r_1$ and using the results from part (a), we see that every right side evaluates to zero.

$$\begin{aligned}
L[e^{r_1t}] &= e^{r_1t}Z(r_1) = 0 \\
L[te^{r_1t}] &= te^{r_1t}Z(r_1) + e^{r_1t}Z'(r_1) = 0 \\
L[t^2e^{r_1t}] &= t^2e^{r_1t}Z(r_1) + 2te^{r_1t}Z'(r_1) + e^{r_1t}Z''(r_1) = 0 \\
L[t^3e^{r_1t}] &= t^3e^{r_1t}Z(r_1) + 3t^2e^{r_1t}Z'(r_1) + 3te^{r_1t}Z''(r_1) + e^{r_1t}Z'''(r_1) = 0 \\
L[t^4e^{r_1t}] &= t^4e^{r_1t}Z(r_1) + 4t^3e^{r_1t}Z'(r_1) + 6t^2e^{r_1t}Z''(r_1) + 4te^{r_1t}Z'''(r_1) + e^{r_1t}Z^{(4)}(r_1) = 0 \\
&\vdots \\
L[t^{s-1}e^{r_1t}] &= t^{s-1}e^{r_1t}Z(r_1) + \cdots + e^{r_1t}Z^{(s-1)}(r_1) = 0
\end{aligned}$$

Therefore, e^{r_1t}, te^{r_1t}, $t^2e^{r_1t}$, $\ldots$, $t^{s-1}e^{r_1t}$ are solutions to Eq. (1).