Problem 11

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y''' - y'' - y' + y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - (r^2e^{rt}) - (re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r^2 - r + 1 = 0$$

$$(r+1)(r-1)^2 = 0$$

$$r = \{-1, 1\}$$

Two solutions to the ODE are then $y = e^{-t}$ and $y = e^t$. By using the method of reduction of order, we can obtain the general solution. Plug in $y(t) = c(t)e^t$ to the ODE.

$$[c(t)e^t]''' - [c(t)e^t]'' - [c(t)e^t]' + [c(t)e^t] = 0$$

Evaluate the derivatives.

$$[c'(t)e^t + c(t)e^t]'' - [c'(t)e^t + c(t)e^t]' - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$[c''(t)e^t + 2c'(t)e^t + c(t)e^t]' - [c''(t)e^t + 2c'(t)e^t + c(t)e^t] - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$[c'''(t)e^t + 3c''(t)e^t + 3c'(t)e^t + c(t)e^t] - [c''(t)e^t + 2c'(t)e^t + c(t)e^t] - [c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

Expand the left side.

$$c'''(t)e^t + 3c''(t)e^t + \cancel{3c'(t)e^t} + \cancel{c(t)e^t} - c''(t)e^t - \cancel{2c'(t)e^t} - \cancel{c(t)e^t} - \cancel{c'(t)e^t} - \cancel{c(t)e^t} + \cancel{c(t)e^t} = 0$$

$$c'''(t)e^t + 2c''(t)e^t = 0$$

Bring the second term to the left side and then divide both sides by $c''(t)e^t$.

$$\frac{c'''(t)}{c''(t)} = -2$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln|c''(t)| = -2$$

An absolute value sign has been included because the logarithm argument cannot be negative. Integrate both sides with respect to t.

$$\ln|c''(t)| = -2t + C_1$$

Exponentiate both sides.

$$\begin{aligned}|c''(t)| &= e^{-2t+C_1}\\ &= e^{C_1}e^{-2t}\end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$c''(t) = \pm e^{C_1}e^{-2t}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$c''(t) = C_2 e^{-2t}$$

Integrate both sides with respect to t again.

$$c'(t) = -\frac{C_2}{2}e^{-2t} + C_3$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{C_2}{4}e^{-2t} + C_3 t + C_4$$

Therefore, since $y(t) = c(t)e^t$,

$$y(t) = C_5 e^{-t} + C_3 t e^t + C_4 e^t,$$

where a new constant C_5 is used for $C_2/4$.