Problem 15

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(6)} + y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt} \quad \rightarrow \quad y^{(6)} = r^6 e^{rt}$$

Substitute these expressions into the ODE.

$$r^6 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^6 + 1 = 0$$

$$r^6 = -1$$

$$r = (-1)^{1/6}$$

$$r = (e^{i\pi})^{1/6}$$

$$r = [e^{i(\pi + 2n\pi)}]^{1/6}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$r = e^{i(\pi/6 + n\pi/3)}$$

The six distinct roots are obtained by setting $n = 0$, $n = 1$, ..., $n = 5$. Other values of n lead to redundant roots.

$$\begin{aligned}
n = 0: &\quad (-1)^{1/6} = e^{i\pi/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\
n = 1: &\quad (-1)^{1/6} = e^{3i\pi/6} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\
n = 2: &\quad (-1)^{1/6} = e^{5i\pi/6} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \\
n = 3: &\quad (-1)^{1/6} = e^{7i\pi/6} = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2} \\
n = 4: &\quad (-1)^{1/6} = e^{9i\pi/6} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i \\
n = 5: &\quad (-1)^{1/6} = e^{11i\pi/6} = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}
\end{aligned}$$

Six solutions to the ODE are then $y = e^{(\sqrt{3}/2 + i/2)t}$ and $y = e^{it}$ and $y = e^{(-\sqrt{3}/2 + i/2)t}$ and $y = e^{(-\sqrt{3}/2 - i/2)t}$ and $y = e^{-it}$ and $y = e^{(\sqrt{3}/2 - i/2)t}$. By the principle of superposition, the general solution is a linear combination of these six.

$$\begin{aligned}
y(t) &= C_1 e^{(\sqrt{3}/2 + i/2)t} + C_2 e^{it} + C_3 e^{(-\sqrt{3}/2 + i/2)t} + C_4 e^{(-\sqrt{3}/2 - i/2)t} + C_5 e^{-it} + C_6 e^{(\sqrt{3}/2 - i/2)t} \\
&= C_1 e^{\sqrt{3}t/2 + it/2} + C_2 e^{it} + C_3 e^{-\sqrt{3}t/2 + it/2} + C_4 e^{-\sqrt{3}t/2 - it/2} + C_5 e^{-it} + C_6 e^{\sqrt{3}t/2 - it/2}
\end{aligned}$$

Use Euler's formula to write the exponential function in terms of sine and cosine.

$$\begin{aligned}
y(t) &= C_1 e^{\sqrt{3}t/2} e^{it/2} + C_2 e^{it} + C_3 e^{-\sqrt{3}t/2} e^{it/2} + C_4 e^{-\sqrt{3}t/2} e^{-it/2} + C_5 e^{-it} + C_6 e^{\sqrt{3}t/2} e^{-it/2} \\
&= e^{\sqrt{3}t/2}(C_1 e^{it/2} + C_6 e^{-it/2}) + e^{-\sqrt{3}t/2}(C_3 e^{it/2} + C_4 e^{-it/2}) + C_2 e^{it} + C_5 e^{-it} \\
&= e^{\sqrt{3}t/2}\left[C_1\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right) + C_6\left(\cos\frac{t}{2} - i\sin\frac{t}{2}\right)\right] \\
&\qquad + e^{-\sqrt{3}t/2}\left[C_3\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right) + C_4\left(\cos\frac{t}{2} - i\sin\frac{t}{2}\right)\right] \\
&\qquad\qquad + C_2(\cos t + i\sin t) + C_5(\cos t - i\sin t) \\
&= e^{\sqrt{3}t/2}\left[(C_1 + C_6)\cos\frac{t}{2} + (iC_1 - iC_6)\sin\frac{t}{2}\right] \\
&\qquad + e^{-\sqrt{3}t/2}\left[(C_3 + C_4)\cos\frac{t}{2} + (iC_3 - iC_4)\sin\frac{t}{2}\right] \\
&\qquad\qquad + (C_2 + C_5)\cos t + (iC_2 - iC_5)\sin t
\end{aligned}$$

Therefore,

$$y(t) = e^{\sqrt{3}t/2}\left(C_7\cos\frac{t}{2} + C_8\sin\frac{t}{2}\right) + e^{-\sqrt{3}t/2}\left(C_9\cos\frac{t}{2} + C_{10}\sin\frac{t}{2}\right) + C_{11}\cos t + C_{12}\sin t.$$