Problem 17

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(6)} - 3y^{(4)} + 3y'' - y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt} \quad \rightarrow \quad y^{(4)} = r^4e^{rt} \quad \rightarrow \quad y^{(6)} = r^6e^{rt}$$

Substitute these expressions into the ODE.

$$r^6e^{rt} - 3(r^4e^{rt}) + 3(r^2e^{rt}) - (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^6 - 3r^4 + 3r^2 - 1 = 0$$

$$(r^2 - 1)^3 = 0$$

$$(r + 1)^3(r - 1)^3 = 0$$

$$r = \{-1, 1\}$$

Two solutions to the ODE are then $y = e^{-t}$ and $y = e^t$. By using the method of reduction of order, we can obtain the general solution. Plug in $y(t) = c(t)e^t$ to the ODE.

$$[c(t)e^t]^{(6)} - 3[c(t)e^t]^{(4)} + 3[c(t)e^t]'' - [c(t)e^t] = 0$$

Evaluate the derivatives.

$$[c'(t)e^t + c(t)e^t]^{(5)} - 3[c'(t)e^t + c(t)e^t]''' + 3[c'(t)e^t + c(t)e^t]' - [c(t)e^t] = 0$$

$$[c''(t)e^t + 2c'(t)e^t + c(t)e^t]^{(4)} - 3[c''(t)e^t + 2c'(t)e^t + c(t)e^t]'' + 3[c''(t)e^t + 2c'(t)e^t + c(t)e^t] - [c(t)e^t] = 0$$

$$\begin{aligned}[c^{(6)}(t)e^t + 6c^{(5)}(t)e^t + 15c^{(4)}(t)e^t + 20c'''(t)e^t + 15c''(t)e^t + 6c'(t)e^t + c(t)e^t]& \\ - 3[c^{(4)}(t)e^t + 4c'''(t)e^t + 6c''(t)e^t + 4c'(t) + c(t)] + 3[c''(t)e^t& \\ + 2c'(t)e^t + c(t)e^t] - [c(t)e^t] = 0&\end{aligned}$$

Simplify the left side.

$$c^{(6)}(t)e^t + 6c^{(5)}(t)e^t + 12c^{(4)}(t)e^t + 8c'''(t)e^t = 0$$

Divide both sides by e^t and make the substitution $b(t) = c'''(t)$.

$$b''' + 6b'' + 12b' + 8b = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $b = e^{st}$.

$$b = e^{st} \quad \rightarrow \quad b' = se^{st} \quad \rightarrow \quad b'' = s^2e^{st} \quad \rightarrow \quad b''' = s^3e^{st}$$

Substitute these expressions into the ODE.

$$s^3e^{st} + 6(s^2e^{st}) + 12(se^{st}) + 8(e^{st}) = 0$$

Divide both sides by e^{st}.

$$s^3 + 6s^2 + 12s + 8 = 0$$

$$(s+2)^3 = 0$$

$$s = \{-2\}$$

One solution to equation (1) is then $b = e^{-2t}$. Apply the method of reduction of order again by plugging in $b(t) = a(t)e^{-2t}$.

$$[a(t)e^{-2t}]''' + 6[a(t)e^{-2t}]'' + 12[a(t)e^{-2t}]' + 8[a(t)e^{-2t}] = 0$$

Evaluate the derivatives.

$$[a'(t)e^{-2t} - 2a(t)e^{-2t}]'' + 6[a'(t)e^{-2t} - 2a(t)e^{-2t}]' + 12[a'(t)e^{-2t} - 2a(t)e^{-2t}] + 8[a(t)e^{-2t}] = 0$$

$$\begin{aligned}[a''(t)e^{-2t} - 4a'(t)e^{-2t} + 4a(t)e^{-2t}]' + 6[a''(t)e^{-2t} - 4a'(t)e^{-2t} + 4a(t)e^{-2t}]& \\ + 12[a'(t)e^{-2t} - 2a(t)e^{-2t}] + 8[a(t)e^{-2t}] &= 0\end{aligned}$$

$$\begin{aligned}[a'''(t)e^{-2t} - 6a''(t)e^{-2t} + 12a'(t)e^{-2t} - 8a(t)e^{-2t}]& \\ + 6[a''(t)e^{-2t} - 4a'(t)e^{-2t} + 4a(t)e^{-2t}] + 12[a'(t)e^{-2t} - 2a(t)e^{-2t}] + 8[a(t)e^{-2t}] &= 0\end{aligned}$$

Simplify the left side.

$$a'''(t)e^{-2t} = 0$$

Multiply both sides by e^{2t}.

$$a'''(t) = 0$$

Integrate both sides with respect to t.

$$a''(t) = C_1$$

Integrate both sides with respect to t a second time.

$$a'(t) = C_1 t + C_2$$

Integrate both sides with respect to t a third time.

$$a(t) = \frac{C_1}{2}t^2 + C_2 t + C_3$$

Since $b(t) = a(t)e^{-2t}$, replace $a(t)$ with $b(t)e^{2t}$.

$$b(t)e^{2t} = \frac{C_1}{2}t^2 + C_2 t + C_3$$

Divide both sides by e^{2t}.

$$b(t) = \frac{C_1}{2}t^2 e^{-2t} + C_2 t e^{-2t} + C_3 e^{-2t}$$

Now replace $b(t)$ with $c'''(t)$.

$$c'''(t) = \frac{C_1}{2}t^2 e^{-2t} + C_2 t e^{-2t} + C_3 e^{-2t}$$

Integrate both sides with respect to t.

$$c''(t) = -\frac{C_1}{8}(2t^2+2t+1)e^{-2t} - \frac{C_2}{4}(2t+1)e^{-2t} - \frac{C_3}{2}e^{-2t} + C_4$$

Integrate both sides with respect to t again.

$$c'(t) = \frac{C_1}{16}(2t^2+4t+3)e^{-2t} + \frac{C_2}{4}(t+1)e^{-2t} + \frac{C_3}{4}e^{-2t} + C_4t + C_5$$

Integrate both sides with respect to t once more.

$$\begin{aligned} c(t) &= -\frac{C_1}{16}(t^2+3t+3)e^{-2t} - \frac{C_2}{16}(2t+3)e^{-2t} - \frac{C_3}{8}e^{-2t} + \frac{C_4}{2}t^2 + C_5t + C_6 \\ &= -\frac{C_1}{16}t^2e^{-2t} + \left(-\frac{3C_1}{16} - \frac{C_2}{8}\right)te^{-2t} + \left(-\frac{3C_1}{16} - \frac{3C_2}{16} - \frac{C_3}{8}\right)e^{-2t} + \frac{C_4}{2}t^2 + C_5t + C_6 \end{aligned}$$

Since $y(t) = c(t)e^t$,

$$y(t) = -\frac{C_1}{16}t^2e^{-t} + \left(-\frac{3C_1}{16} - \frac{C_2}{8}\right)te^{-t} + \left(-\frac{3C_1}{16} - \frac{3C_2}{16} - \frac{C_3}{8}\right)e^{-t} + \frac{C_4}{2}t^2e^t + C_5te^t + C_6e^t.$$

Therefore,

$$y(t) = C_7t^2e^{-t} + C_8te^{-t} + C_9e^{-t} + C_{10}t^2e^t + C_5te^t + C_6e^t.$$