Problem 18

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(6)} - y'' = 0$$

Solution

Start by making the substitution $u = y''$.

$$u^{(4)} - u = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt} \quad \rightarrow \quad u''' = r^3 e^{rt} \quad \rightarrow \quad u^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - 1 = 0$$
$$(r^2 - 1)(r^2 + 1) = 0$$
$$r = \{-1, 1, -i, i\}$$

Four solutions to the ODE are then $y = e^{-t}$ and $y = e^t$ and $y = e^{it}$ and $y = e^{-it}$. By the principle of superposition, the general solution for u is a linear combination of these four.

$$u(t) = C_1 e^{-t} + C_2 e^t + C_3 e^{it} + C_4 e^{-it}$$

Change back to y now.

$$y''(t) = C_1 e^{-t} + C_2 e^t + C_3 e^{it} + C_4 e^{-it}$$

Integrate both sides with respect to t.

$$y'(t) = -C_1 e^{-t} + C_2 e^t + \frac{C_3}{i} e^{it} - \frac{C_4}{i} e^{-it} + C_5$$

Integrate both sides with respect to t once more.

$$\begin{aligned} y(t) &= C_1 e^{-t} + C_2 e^t + \frac{C_3}{i^2} e^{it} + \frac{C_4}{i^2} e^{-it} + C_5 t + C_6 \\ &= C_1 e^{-t} + C_2 e^t + \frac{C_3}{i^2}(\cos t + i \sin t) + \frac{C_4}{i^2}(\cos t - i \sin t) + C_5 t + C_6 \\ &= C_1 e^{-t} + C_2 e^t + \left(\frac{C_3}{i^2} + \frac{C_4}{i^2}\right) \cos t + \left(\frac{C_3}{i} - \frac{C_4}{i}\right) \sin t + C_5 t + C_6 \end{aligned}$$

Therefore, using new constants for the quantities in parentheses,

$$y(t) = C_1 e^{-t} + C_2 e^t + C_7 \cos t + C_8 \sin t + C_5 t + C_6.$$