Problem 19

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$$

Solution

Start by making the substitution $u = y'$.

$$u^{(4)} - 3u''' + 3u'' - 3u' + 2u = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2e^{rt} \quad \rightarrow \quad u''' = r^3e^{rt} \quad \rightarrow \quad u^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - 3(r^3e^{rt}) + 3(r^2e^{rt}) - 3(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 - 3r^3 + 3r^2 - 3r + 2 = 0$$

$$(r-1)(r-2)(r^2+1) = 0$$

$$r = \{1, 2, -i, i\}$$

Four solutions to the ODE are then $y = e^t$ and $y = e^{2t}$ and $y = e^{it}$ and $y = e^{-it}$. By the principle of superposition, the general solution for u is a linear combination of these four.

$$u(t) = C_1e^t + C_2e^{2t} + C_3e^{it} + C_4e^{-it}$$

Change back to y now.

$$y'(t) = C_1e^t + C_2e^{2t} + C_3e^{it} + C_4e^{-it}$$

Integrate both sides with respect to t.

$$\begin{aligned} y(t) &= C_1e^t + \frac{C_2}{2}e^{2t} + \frac{C_3}{i}e^{it} - \frac{C_4}{i}e^{-it} + C_5 \\ &= C_1e^t + \frac{C_2}{2}e^{2t} + \frac{C_3}{i}(\cos t + i\sin t) - \frac{C_4}{i}(\cos t - i\sin t) + C_5 \\ &= C_1e^t + \frac{C_2}{2}e^{2t} + \left(\frac{C_3}{i} - \frac{C_4}{i}\right)\cos t + (C_3 + C_4)\sin t + C_5 \end{aligned}$$

Therefore, using new constants,

$$y(t) = C_1e^t + C_6e^{2t} + C_7\cos t + C_8\sin t + C_5.$$