Problem 22

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$y^{(4)} + 2y'' + y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt} \quad \to \quad y^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} + 2(r^2 e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$\begin{gathered} r^4 + 2r^2 + 1 = 0 \\ (r^2+1)^2 = 0 \qquad (1) \\ r^2 + 1 = 0 \\ r = \{-i, i\} \end{gathered}$$

Two solutions to the ODE are then $y = e^{-it}$ and $y = e^{it}$. The multiplicity of each of these roots is 2 because the quantity in equation (1) is squared. That means a second linearly independent solution can be obtained from each of these two by including a factor of t: $y = te^{-it}$ and $y = te^{it}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$\begin{aligned} y(t) &= C_1 e^{-it} + C_2 e^{it} + C_3 t e^{-it} + C_4 t e^{it} \\ &= C_1(\cos t - i \sin t) + C_2(\cos t + i \sin t) + C_3 t(\cos t - i \sin t) + C_4 t(\cos t + i \sin t) \\ &= (C_1 + C_2)\cos t + (C_3 + C_4) t \cos t + (-iC_1 + iC_2)\sin t + (-iC_3 + iC_4) t \sin t \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = C_5 \cos t + C_6 t \cos t + C_7 \sin t + C_8 t \sin t.$$