Problem 27

In each of Problems 11 through 28, find the general solution of the given differential equation.

$$12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt} \quad \rightarrow \quad y^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$12(r^4e^{rt}) + 31(r^3e^{rt}) + 75(r^2e^{rt}) + 37(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$$

$$(3r+1)(4r+1)(r^2+2r+5) = 0$$

Use the zero product theorem.

$$\begin{aligned} 3r+1=0 \quad &\text{or} \quad 4r+1=0 \quad \text{or} \quad r^2+2r+5=0 \\ r = -\frac{1}{3} \quad &\text{or} \quad r = -\frac{1}{4} \quad \text{or} \quad r = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i \end{aligned}$$

$$r = \left\{-\frac{1}{3}, -\frac{1}{4}, -1-2i, -1+2i\right\}$$

Four solutions to the ODE are then $y = e^{-t/3}$ and $y = e^{-t/4}$ and $y = e^{(-1-2i)t}$ and $y = e^{(-1+2i)t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$\begin{aligned} y(t) &= C_1e^{-t/3} + C_2e^{-t/4} + C_3e^{(-1-2i)t} + C_4e^{(-1+2i)t} \\ &= C_1e^{-t/3} + C_2e^{-t/4} + C_3e^{-t-2it} + C_4e^{-t+2it} \\ &= C_1e^{-t/3} + C_2e^{-t/4} + C_3e^{-t}e^{-2it} + C_4e^{-t}e^{2it} \\ &= C_1e^{-t/3} + C_2e^{-t/4} + C_3e^{-t}(\cos 2t - i\sin 2t) + C_4e^{-t}(\cos 2t + i\sin 2t) \\ &= C_1e^{-t/3} + C_2e^{-t/4} + e^{-t}(C_3 + C_4)\cos 2t + e^{-t}(-iC_3 + iC_4)\sin 2t \end{aligned}$$

Therefore, using new arbitrary constants,

$$y(t) = C_1e^{-t/3} + C_2e^{-t/4} + e^{-t}C_5\cos 2t + e^{-t}C_6\sin 2t.$$