# Problem 29

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y''' + y' = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$$

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### Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} + re^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^3 + r = 0$$

$$r(r^2 + 1) = 0$$

$$r = \{0, -i, i\}$$

Three solutions to the ODE are then $y = e^0 = 1$ and $y = e^{-it}$ and $y = e^{it}$. By the principle of superposition, the general solution for $y$ is a linear combination of these three.

$$\begin{aligned} y(t) &= C_1 + C_2 e^{-it} + C_3 e^{it} \\ &= C_1 + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \\ &= C_1 + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\ &= C_1 + C_4 \cos t + C_5 \sin t \end{aligned}$$

Differentiate this solution twice with respect to $t$.

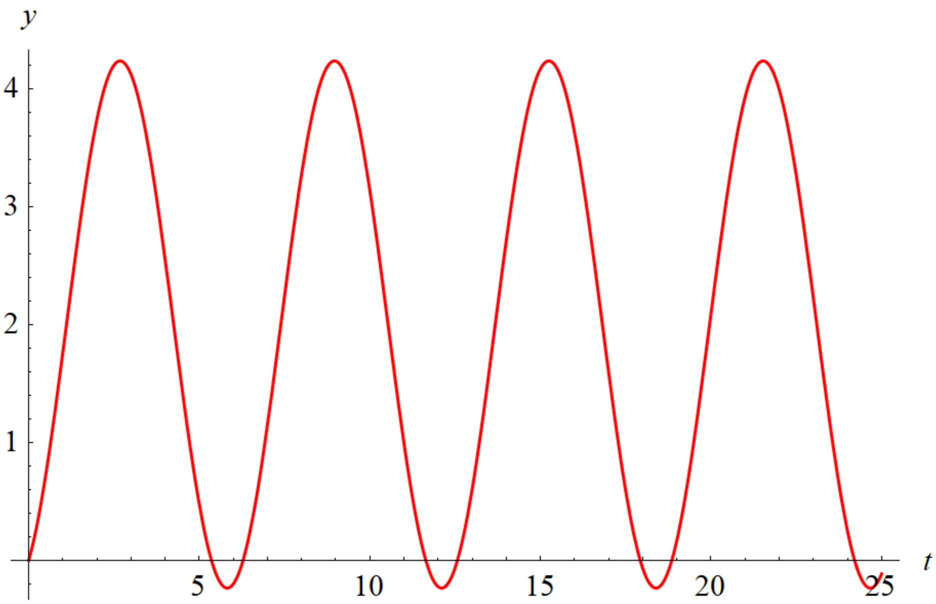
$$\begin{aligned} y'(t) &= -C_4 \sin t + C_5 \cos t \\ y''(t) &= -C_4 \cos t - C_5 \sin t \end{aligned}$$

Apply the initial conditions now to determine $C_1$, $C_4$, and $C_5$.

$$\begin{aligned} y(0) &= C_1 + C_4 = 0 \\ y'(0) &= C_5 = 1 \\ y''(0) &= -C_4 = 2 \end{aligned}$$

Solving this system yields $C_1 = 2$, $C_4 = -2$, and $C_5 = 1$. Therefore,

$$y(t) = 2 - 2\cos t + \sin t.$$




The solution keeps oscillating with the same amplitude and period for all $t$.