Problem 30

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y^{(4)} + y = 0; \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt} \quad \to \quad y^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 + 1 = 0$$
$$r^4 = -1$$
$$r = (-1)^{1/4}$$
$$r = (1e^{i\pi})^{1/4}$$
$$r = [e^{i(\pi + 2n\pi)}]^{1/4}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$r = e^{i(\pi/4 + n\pi/2)}$$

The four distinct roots can be obtained by setting $n = 0$, $n = 1$, $n = 2$, and $n = 3$. Other values of n lead to redundant roots.

$$n = 0: \quad e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$
$$n = 1: \quad e^{3i\pi/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$
$$n = 2: \quad e^{5i\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$
$$n = 3: \quad e^{7i\pi/4} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Four solutions to the ODE are then $y = e^{(1/\sqrt{2} + i/\sqrt{2})t}$ and $y = e^{(-1/\sqrt{2} + i/\sqrt{2})t}$ and $y = e^{(-1/\sqrt{2} - i/\sqrt{2})t}$ and $y = e^{(1/\sqrt{2} - i/\sqrt{2})t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$y(t) = C_1 e^{(1/\sqrt{2} + i/\sqrt{2})t} + C_2 e^{(-1/\sqrt{2} + i/\sqrt{2})t} + C_3 e^{(-1/\sqrt{2} - i/\sqrt{2})t} + C_4 e^{(1/\sqrt{2} - i/\sqrt{2})t}$$

Differentiate this solution three times with respect to t.

$$\begin{aligned}
y'(t) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)e^{(1/\sqrt{2}+i/\sqrt{2})t}+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)e^{(-1/\sqrt{2}+i/\sqrt{2})t}\\
&\quad+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)e^{(-1/\sqrt{2}-i/\sqrt{2})t}+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)e^{(1/\sqrt{2}-i/\sqrt{2})t}\\
y''(t) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2e^{(1/\sqrt{2}+i/\sqrt{2})t}+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2e^{(-1/\sqrt{2}+i/\sqrt{2})t}\\
&\quad+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^2e^{(-1/\sqrt{2}-i/\sqrt{2})t}+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^2e^{(1/\sqrt{2}-i/\sqrt{2})t}\\
y'''(t) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^3e^{(1/\sqrt{2}+i/\sqrt{2})t}+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^3e^{(-1/\sqrt{2}+i/\sqrt{2})t}\\
&\quad+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^3e^{(-1/\sqrt{2}-i/\sqrt{2})t}+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^3e^{(1/\sqrt{2}-i/\sqrt{2})t}
\end{aligned}$$

Apply the initial conditions now to determine C_1, C_2, C_3, and C_4.

$$\begin{aligned}
y(0) &= C_1+C_2+C_3+C_4=0\\
y'(0) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=0\\
y''(0) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^2+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^2=-1\\
y'''(0) &= C_1\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^3+C_2\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^3+C_3\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^3+C_4\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^3=0
\end{aligned}$$

Solving this system of equations yields

$$\begin{aligned}
C_1 &= \frac{i}{4}\\
C_2 &= -\frac{i}{4}\\
C_3 &= \frac{i}{4}\\
C_4 &= -\frac{i}{4}
\end{aligned}$$

Now that the constants are known, write $y(t)$ in terms of real functions.

$$\begin{aligned}
y(t) &= C_1e^{t/\sqrt{2}+it/\sqrt{2}}+C_2e^{-t/\sqrt{2}+it/\sqrt{2}}+C_3e^{-t/\sqrt{2}-it/\sqrt{2}}+C_4e^{t/\sqrt{2}-it/\sqrt{2}}\\
&= C_1e^{t/\sqrt{2}}e^{it/\sqrt{2}}+C_2e^{-t/\sqrt{2}}e^{it/\sqrt{2}}+C_3e^{-t/\sqrt{2}}e^{-it/\sqrt{2}}+C_4e^{t/\sqrt{2}}e^{-it/\sqrt{2}}\\
&= e^{t/\sqrt{2}}(C_1e^{it/\sqrt{2}}+C_4e^{-it/\sqrt{2}})+e^{-t/\sqrt{2}}(C_2e^{it/\sqrt{2}}+C_3e^{-it/\sqrt{2}})\\
&= e^{t/\sqrt{2}}\left[C_1\left(\cos\frac{t}{\sqrt{2}}+i\sin\frac{t}{\sqrt{2}}\right)+C_4\left(\cos\frac{t}{\sqrt{2}}-i\sin\frac{t}{\sqrt{2}}\right)\right]\\
&\quad+e^{-t/\sqrt{2}}\left[C_2\left(\cos\frac{t}{\sqrt{2}}+i\sin\frac{t}{\sqrt{2}}\right)+C_3\left(\cos\frac{t}{\sqrt{2}}-i\sin\frac{t}{\sqrt{2}}\right)\right]
\end{aligned}$$

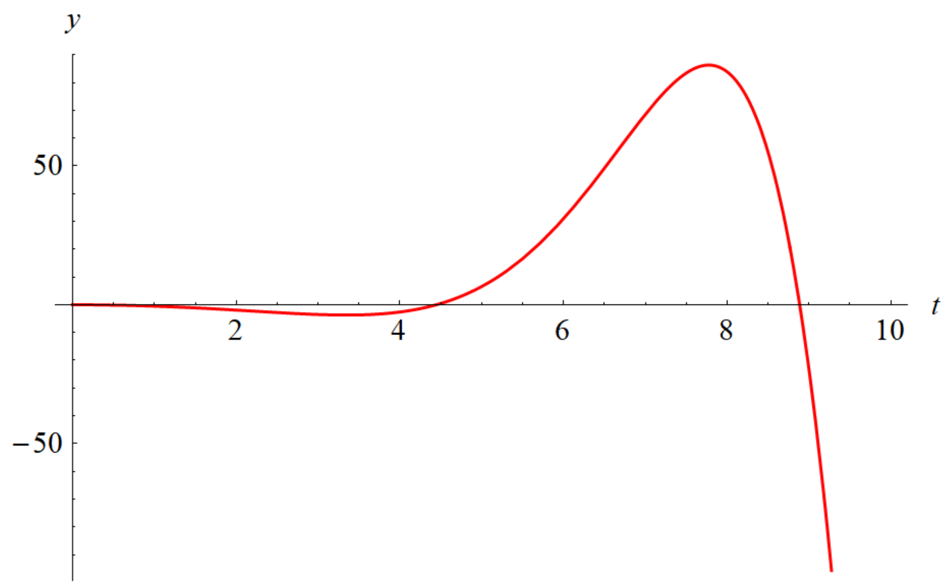
$$y(t) = e^{t/\sqrt{2}}\left[(C_1 + C_4)\cos\frac{t}{\sqrt{2}} + (iC_1 - iC_4)\sin\frac{t}{\sqrt{2}}\right] + e^{-t/\sqrt{2}}\left[(C_2 + C_3)\cos\frac{t}{\sqrt{2}} + (iC_2 - iC_3)\sin\frac{t}{\sqrt{2}}\right]$$

Evaluate the constants.

$$y(t) = e^{t/\sqrt{2}}\left[(0)\cos\frac{t}{\sqrt{2}} + \left(-\frac{1}{2}\right)\sin\frac{t}{\sqrt{2}}\right] + e^{-t/\sqrt{2}}\left[(0)\cos\frac{t}{\sqrt{2}} + \left(\frac{1}{2}\right)\sin\frac{t}{\sqrt{2}}\right]$$

Therefore,

$$y(t) = \frac{1}{2}e^{-t/\sqrt{2}}\sin\frac{t}{\sqrt{2}} - \frac{1}{2}e^{t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}.$$



In the limit as $t \to \infty$, the first term in $y(t)$ tends to zero because of the exponential function; the second term, however, makes $y(t)$ diverge.