Problem 31

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y^{(4)} - 4y''' + 4y'' = 0; \qquad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$$

Solution

Start by making the substitution $u = y''$.

$$u'' - 4u' + 4 = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \to \quad u' = re^{rt} \quad \to \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0 \qquad (1) \\ r &= \{2\} \end{aligned}$$

One solution to the ODE is then $u = e^{2t}$. Because the quantity in equation (1) is squared, the multiplicity of the $r = 2$ root is 2. That means a second linearly independent solution can be obtained by including a factor of t in the first: $u = te^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$u(t) = C_1 e^{2t} + C_2 te^{2t}$$

Change back to y now.

$$y''(t) = C_1 e^{2t} + C_2 te^{2t}$$

Integrate both sides with respect to t.

$$y'(t) = \frac{C_1}{2} e^{2t} + \frac{C_2}{4}(2t-1)e^{2t} + C_3$$

Integrate both sides with respect to t once more.

$$\begin{aligned} y(t) &= \frac{C_1}{4} e^{2t} + \frac{C_2}{4}(t-1)e^{2t} + C_3 t + C_4 \\ &= \frac{C_1}{4} e^{2t} + \frac{C_2}{4} te^{2t} - \frac{C_2}{4} e^{2t} + C_3 t + C_4 \\ &= \left(\frac{C_1}{4} - \frac{C_2}{4}\right) e^{2t} + \frac{C_2}{4} te^{2t} + C_3 t + C_4 \end{aligned}$$

Therefore, using new arbitrary constants, the general solution is

$$y(t) = C_5 e^{2t} + C_6 te^{2t} + C_3 t + C_4.$$

Differentiate it three times with respect to t.

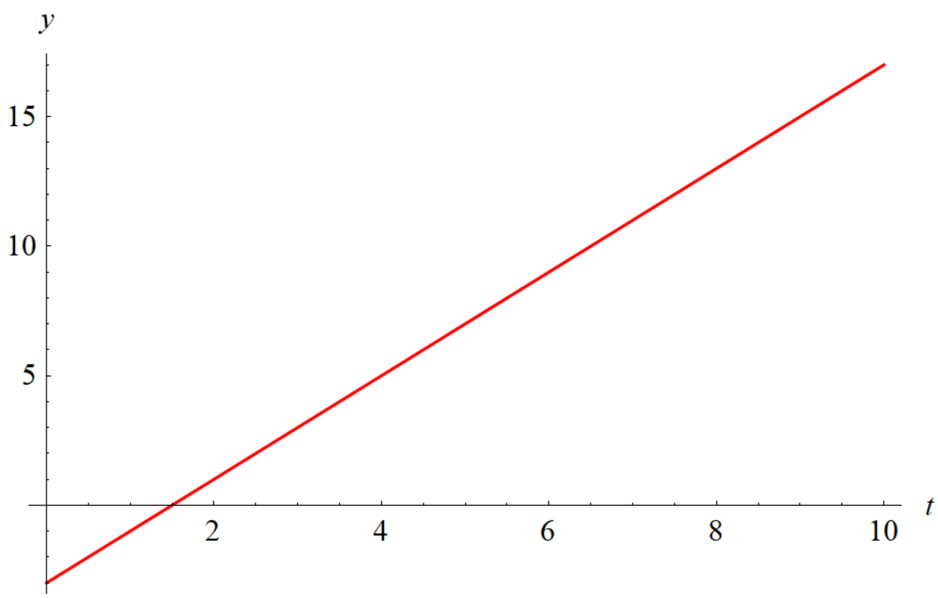
$$y'(t) = 2C_5e^{2t} + C_6(2t+1)e^{2t} + C_3$$
$$y''(t) = 4C_5e^{2t} + 4C_6(t+1)e^{2t}$$
$$y'''(t) = 8C_5e^{2t} + 4C_6(2t+3)e^{2t}$$

Now apply the initial conditions to determine C_5, C_6, C_3, and C_4.

$$y(1) = C_5e^2 + C_6e^2 + C_3 + C_4 = -1$$
$$y'(1) = 2C_5e^2 + 3C_6e^2 + C_3 = 2$$
$$y''(1) = 4C_5e^2 + 8C_6e^2 = 0$$
$$y'''(1) = 8C_5e^2 + 20C_6e^2 = 0$$

Solving this system of equations yields $C_5 = 0$, $C_6 = 0$, $C_3 = 2$, and $C_4 = -3$. Therefore,

$$y(t) = 2t - 3.$$



In the limit as $t \to \infty$, the solution goes to ∞.