Problem 36

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0; \qquad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0, \quad y'''(0) = 3$$

Solution

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2e^{rt} \quad \to \quad y''' = r^3e^{rt} \quad \to \quad y^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 6(r^3e^{rt}) + 17(r^2e^{rt}) + 22(re^{rt}) + 14(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$$

$$(r^2 + 2r + 2)(r^2 + 4r + 7) = 0$$

Use the zero product theorem.

$$\begin{aligned} r^2 + 2r + 2 &= 0 & &\text{or} & r^2 + 4r + 7 &= 0 \\ r &= \frac{-2 \pm \sqrt{4 - 4(2)}}{2} & &\text{or} & r &= \frac{-4 \pm \sqrt{16 - 4(7)}}{2} \\ r &= \frac{-2 \pm \sqrt{-4}}{2} & &\text{or} & r &= \frac{-4 \pm \sqrt{-12}}{2} \\ r &= \frac{-2 \pm 2i}{2} & &\text{or} & r &= \frac{-4 \pm 2i\sqrt{3}}{2} \\ r &= -1 \pm i & &\text{or} & r &= -2 \pm i\sqrt{3} \end{aligned}$$

$$r = \{-1 - i, -1 + i, -2 - i\sqrt{3}, -2 + i\sqrt{3}\}$$

Four solutions to the ODE are then $y = e^{(-1-i)t}$ and $y = e^{(-1+i)t}$ and $y = e^{(-2-i\sqrt{3})t}$ and $y = e^{(-2+i\sqrt{3})t}$. By the principle of superposition, the general solution for y is a linear combination of these four.

$$y(t) = C_1e^{(-1-i)t} + C_2e^{(-1+i)t} + C_3e^{(-2-i\sqrt{3})t} + C_4e^{(-2+i\sqrt{3})t}$$

Differentiate it with respect to t three times.

$$y'(t) = C_1(-1 - i)e^{(-1-i)t} + C_2(-1 + i)e^{(-1+i)t} + C_3(-2 - i\sqrt{3})e^{(-2-i\sqrt{3})t} + C_4(-2 + i\sqrt{3})e^{(-2+i\sqrt{3})t}$$

$$y''(t) = C_1(-1 - i)^2e^{(-1-i)t} + C_2(-1 + i)^2e^{(-1+i)t} + C_3(-2 - i\sqrt{3})^2e^{(-2-i\sqrt{3})t} + C_4(-2 + i\sqrt{3})^2e^{(-2+i\sqrt{3})t}$$

$$y'''(t) = C_1(-1 - i)^3e^{(-1-i)t} + C_2(-1 + i)^3e^{(-1+i)t} + C_3(-2 - i\sqrt{3})^3e^{(-2-i\sqrt{3})t} + C_4(-2 + i\sqrt{3})^3e^{(-2+i\sqrt{3})t}$$

Apply the initial conditions now to determine C_1, C_2, C_3, and C_4.

$$\begin{aligned} y(0) &= C_1 + C_2 + C_3 + C_4 = 1 \\ y'(0) &= C_1(-1 - i) + C_2(-1 + i) + C_3(-2 - i\sqrt{3}) + C_4(-2 + i\sqrt{3}) = -2 \\ y''(0) &= C_1(-1 - i)^2 + C_2(-1 + i)^2 + C_3(-2 - i\sqrt{3})^2 + C_4(-2 + i\sqrt{3})^2 = 0 \\ y'''(0) &= C_1(-1 - i)^3 + C_2(-1 + i)^3 + C_3(-2 - i\sqrt{3})^3 + C_4(-2 + i\sqrt{3})^3 = 3 \end{aligned}$$

Solving this system of equations yields

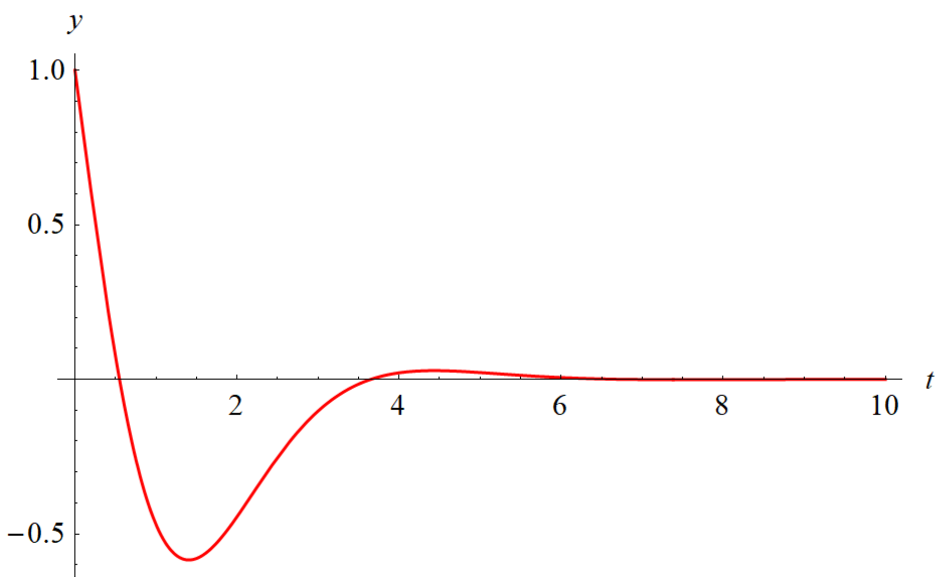
$$C_1 = \frac{21}{26} - i\frac{19}{13}$$
$$C_2 = \frac{21}{26} + i\frac{19}{13}$$
$$C_3 = -\frac{4}{13} + i\frac{17\sqrt{3}}{78}$$
$$C_4 = -\frac{4}{13} - i\frac{17\sqrt{3}}{78}.$$

Now that the constants are known, write $y(t)$ in terms of real functions.

$$\begin{aligned} y(t) &= C_1e^{-t-it} + C_2e^{-t+it} + C_3e^{-2t-i\sqrt{3}t} + C_4e^{-2t+i\sqrt{3}t} \\ &= C_1e^{-t}e^{-it} + C_2e^{-t}e^{it} + C_3e^{-2t}e^{-i\sqrt{3}t} + C_4e^{-2t}e^{i\sqrt{3}t} \\ &= e^{-t}(C_1e^{-it} + C_2e^{it}) + e^{-2t}(C_3e^{-i\sqrt{3}t} + C_4e^{i\sqrt{3}t}) \\ &= e^{-t}[C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t)] + e^{-2t}[C_3(\cos\sqrt{3}t - i\sin\sqrt{3}t) + C_4(\cos\sqrt{3}t + i\sin\sqrt{3}t)] \\ &= e^{-t}[(C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t] + e^{-2t}[(C_3 + C_4)\cos\sqrt{3}t + (-iC_3 + iC_4)\sin\sqrt{3}t] \end{aligned}$$

Therefore, evaluating the constants,

$$y(t) = e^{-t}\left(\frac{21}{13}\cos t - \frac{38}{13}\sin t\right) + e^{-2t}\left(-\frac{8}{13}\cos\sqrt{3}t + \frac{17}{13\sqrt{3}}\sin\sqrt{3}t\right).$$



Because of the decaying exponential functions, $y(t) \to 0$ as $t \to \infty$.