Problem 40

In this problem we outline one way to show that if $r_1, \ldots, r_n$ are all real and different, then $e^{r_1 t}$, $\ldots$, $e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty \tag{i}$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to t, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2-r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n-r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n - r_{n-1}) \cdots (r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

Hence $c_n = 0$, and therefore,

$$c_1 e^{r_1 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0.$$

(d) Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \cdots = c_1 = 0$. Thus the functions $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent.

Solution

Consider the linear relation in Eq. (i).

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t} + c_4 e^{r_4 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty$$

Multiply both sides by $e^{-r_1 t}$.

$$c_1 + c_2 e^{(r_2-r_1)t} + c_3 e^{(r_3-r_1)t} + c_4 e^{(r_4-r_1)t} + \cdots + c_n e^{(r_n-r_1)t} = 0$$

Differentiate both sides with respect to t.

$$c_2(r_2 - r_1)e^{(r_2-r_1)t} + c_3(r_3 - r_1)e^{(r_3-r_1)t} + c_4(r_4 - r_1)e^{(r_4-r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n-r_1)t} = 0$$

Multiply both sides by $e^{-(r_2-r_1)t}$.

$$c_2(r_2 - r_1) + c_3(r_3 - r_1)e^{(r_3-r_2)t} + c_4(r_4 - r_1)e^{(r_4-r_2)t} + \cdots + c_n(r_n - r_1)e^{(r_n-r_2)t} = 0$$

Differentiate both sides with respect to t.

$$c_3(r_3 - r_1)(r_3 - r_2)e^{(r_3-r_2)t} + c_4(r_4 - r_1)(r_4 - r_2)e^{(r_4-r_2)t} + \cdots + c_n(r_n - r_1)(r_n - r_2)e^{(r_n-r_2)t} = 0$$

Multiply both sides by $e^{-(r_3-r_2)t}$.

$$c_3(r_3 - r_1)(r_3 - r_2) + c_4(r_4 - r_1)(r_4 - r_2)e^{(r_4-r_3)t} + \cdots + c_n(r_n - r_1)(r_n - r_2)e^{(r_n-r_3)t} = 0$$

Differentiate both sides with respect to t.

$$c_4(r_4-r_1)(r_4-r_2)(r_4-r_3)e^{(r_4-r_3)t}+\cdots+c_n(r_n-r_1)(r_n-r_2)(r_n-r_3)e^{(r_n-r_3)t}=0$$

Multiply both sides by $e^{-(r_4-r_3)t}$.

$$c_4(r_4-r_1)(r_4-r_2)(r_4-r_3)+\cdots+c_n(r_n-r_1)(r_n-r_2)(r_n-r_3)e^{(r_n-r_4)t}=0$$

Differentiate both sides with respect to t.

$$\cdots+c_n(r_n-r_1)(r_n-r_2)(r_n-r_3)(r_n-r_4)e^{(r_n-r_4)t}=0$$

Continue in the same manner until only the last term remains.

$$c_n(r_n-r_1)(r_n-r_2)(r_n-r_3)(r_n-r_4)\cdots(r_n-r_{n-1})e^{(r_n-r_{n-1})t}=0$$

Consequently, $c_n=0$, and Eq. (i) reduces to

$$c_1e^{r_1t}+c_2e^{r_2t}+c_3e^{r_3t}+c_4e^{r_4t}+\cdots+c_{n-1}e^{r_{n-1}t}=0$$

Multiply both sides by e^{-r_1t}.

$$c_1+c_2e^{(r_2-r_1)t}+c_3e^{(r_3-r_1)t}+c_4e^{(r_4-r_1)t}+\cdots+c_{n-1}e^{(r_{n-1}-r_1)t}=0$$

Differentiate both sides with respect to t.

$$c_2(r_2-r_1)e^{(r_2-r_1)t}+c_3(r_3-r_1)e^{(r_3-r_1)t}+c_4(r_4-r_1)e^{(r_4-r_1)t}+\cdots+c_{n-1}(r_{n-1}-r_1)e^{(r_{n-1}-r_1)t}=0$$

Multiply both sides by $e^{-(r_2-r_1)t}$.

$$c_2(r_2-r_1)+c_3(r_3-r_1)e^{(r_3-r_2)t}+c_4(r_4-r_1)e^{(r_4-r_2)t}+\cdots+c_{n-1}(r_{n-1}-r_1)e^{(r_{n-1}-r_2)t}=0$$

Differentiate both sides with respect to t.

$$c_3(r_3-r_1)(r_3-r_2)e^{(r_3-r_2)t}+c_4(r_4-r_1)(r_4-r_2)e^{(r_4-r_2)t}+\cdots+c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)e^{(r_{n-1}-r_2)t}=0$$

Multiply both sides by $e^{-(r_3-r_2)t}$.

$$c_3(r_3-r_1)(r_3-r_2)+c_4(r_4-r_1)(r_4-r_2)e^{(r_4-r_3)t}+\cdots+c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)e^{(r_{n-1}-r_3)t}=0$$

Differentiate both sides with respect to t.

$$c_4(r_4-r_1)(r_4-r_2)(r_4-r_3)e^{(r_4-r_3)t}+\cdots+c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)(r_{n-1}-r_3)e^{(r_{n-1}-r_3)t}=0$$

Multiply both sides by $e^{-(r_4-r_3)t}$.

$$c_4(r_4-r_1)(r_4-r_2)(r_4-r_3)+\cdots+c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)(r_{n-1}-r_3)e^{(r_{n-1}-r_4)t}=0$$

Differentiate both sides with respect to t.

$$\cdots+c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)(r_{n-1}-r_3)(r_{n-1}-r_4)e^{(r_{n-1}-r_4)t}=0$$

Continue in the same manner until only the last term remains.

$$c_{n-1}(r_{n-1}-r_1)(r_{n-1}-r_2)(r_{n-1}-r_3)(r_{n-1}-r_4)\cdots(r_{n-1}-r_{n-2})e^{(r_{n-1}-r_{n-2})t}=0$$

Consequently, $c_{n-1}=0$. Similarly, c_{n-2}, c_{n-3}, $\ldots$, c_3, c_2, and c_1 are zero as well. Therefore, the functions, $e^{r_1t},\ldots,e^{r_nt}$, are linearly independent.