# Problem 40

In this problem we outline one way to show that if $r_1, \ldots, r_n$ are all real and different, then $e^{r_1 t}$, $\ldots$, $e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty \tag{i}$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to $t$, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2-r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n-r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to $t$ to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n - r_{n-1}) \cdots (r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

Hence $c_n = 0$, and therefore,

$$c_1 e^{r_1 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0.$$

(d) Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \cdots = c_1 = 0$. Thus the functions $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent.