Problem 41

In this problem we indicate one way to show that if $r = r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1 t}$, $te^{r_1 t}$, …, $t^{s-1}e^{r_1 t}$ are solutions of Eq. (1). This problem extends to nth order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{rt}] = e^{rt}Z(r) \tag{i}$$

and differentiate repeatedly with respect to r, setting $r = r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s, then $Z(r) = (r - r_1)^s q(r)$, where $q(r)$ is a polynomial of degree $n - s$ and $q(r_1) \neq 0$. Show that $Z(r_1)$, $Z'(r_1)$, …, $Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1) \neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r, show that

$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}],$$
$$\vdots$$
$$\frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}] = L[t^{s-1}e^{rt}]$$

(c) Show that $e^{r_1 t}$, $te^{r_1 t}$,…, $t^{s-1}e^{r_1 t}$ are solutions of Eq. (1).