Problem 5

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y^{(4)} - 4y'' = t^2 + e^t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - 4y_c'' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - 4(r^2e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 - 4r^2$$

$$r^2(r^2 - 4) = 0$$

$$r = \{-2, 0, 2\}$$

Three solutions to equation (1) are then $y_c = e^{-2t}$ and $y_c = e^0 = 1$ and $y_c = e^{2t}$. Since the $r = 0$ root has a multiplicity of 2, a second linearly independent solution can be obtained by including a factor of t: $y_c = te^0 = t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$y_c(t) = C_1e^{-2t} + C_2 + C_3t + C_4e^{2t}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - 4y_p'' = t^2 + e^t.$$

The right side has two terms. To account for the first one, we would include $A + Bt + Ct^2$ in the trial solution, but because 1 and t already satisfies y_c, an extra factor of t^2 is needed. To account for the second one, we will include De^t in the trial solution. Substitute $y_p(t) = t^2(A + Bt + Ct^2) + De^t$ in the ODE to determine A and B and C and D.

$$[t^2(A + Bt + Ct^2) + De^t]^{(4)} - 4[t^2(A + Bt + Ct^2) + De^t]'' = t^2 + e^t$$

Evaluate the derivatives.

$$(24C + De^t) - 4(2A + 6Bt + 12Ct^2 + De^t) = t^2 + e^t$$

Simplify the left side.

$$24C - 8A - 24Bt - 48Ct^2 - 3De^t = t^2 + e^t$$

Match the coefficients to obtain a system of equations for A, B, C, and D.

$$\begin{aligned} 24C - 8A &= 0 \\ -24B &= 0 \\ -48C &= 1 \\ -3D &= 1 \end{aligned}$$

Solving this system yields $A = -1/16$, $B = 0$, $C = -1/48$, and $D = -1/3$. As a result, the particular solution is $y_p(t) = t^2(-1/16 - t^2/48) - (1/3)e^t$, and the general solution is

$$y(t) = C_1 e^{-2t} + C_2 + C_3 t + C_4 e^{2t} - \frac{1}{3}e^t - \frac{t^2}{16} - \frac{t^4}{48}.$$