Problem 6

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y^{(4)} + 2y'' + y = 3 + \cos 2t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c'' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 2(r^2e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 + 2r^2 + 1 = 0$$
$$(r^2 + 1)^2 = 0$$
$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of t: $y_c = te^{-it}$ and $y_c = te^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned}
y_c(t) &= C_1e^{-it} + C_2e^{it} + C_3te^{-it} + C_4te^{it} \\
&= C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t) + C_3t(\cos t - i\sin t) + C_4t(\cos t + i\sin t) \\
&= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + t(C_3 + C_4)\cos t + t(-iC_3 + iC_4)\sin t \\
&= C_5\cos t + C_6\sin t + C_7t\cos t + C_8t\sin t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p'' + y_p = 3 + \cos 2t.$$

The right side has two terms. To account for the first one, we will include A in the trial solution. To account for the second one, we will include $B\cos 2t$ in the trial solution. Substitute $y_p(t) = A + B\cos 2t$ in the ODE to determine A and B.

$$(A + B\cos 2t)^{(4)} + 2(A + B\cos 2t)'' + (A + B\cos 2t) = 3 + \cos 2t$$

Evaluate the derivatives.

$$(16B\cos 2t) + 2(-4B\cos 2t) + (A + B\cos t) = 3 + \cos 2t$$

Simplify the left side.

$$A + 9B\cos 2t = 3 + \cos 2t$$

Match the coefficients to obtain a system of equations for A and B.

$$\begin{aligned} A &= 3 \\ 9B &= 1 \end{aligned}$$

Solving this system yields $A = 3$ and $B = 1/9$. As a result, the particular solution is $y_p(t) = 3 + (1/9)\cos 2t$, and the general solution is

$$y(t) = C_5\cos t + C_6\sin t + C_7 t\cos t + C_8 t\sin t + 3 + \frac{1}{9}\cos 2t.$$