Problem 7

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y^{(6)} + y''' = t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(6)} + y_c''' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(6)} = r^6e^{rt}$$

Substitute these expressions into the ODE.

$$r^6e^{rt} + r^3e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^6 + r^3 = 0$$

$$r^3(r+1)(r^2 - r + 1) = 0$$

Use the zero product theorem.

$$r^3 = 0 \qquad \text{or} \qquad r + 1 = 0 \qquad \text{or} \qquad r^2 - r + 1 = 0$$

$$r = 0 \qquad \text{or} \qquad r = -1 \qquad \text{or} \qquad r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$r = \left\{-1, 0, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}$$

Four solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^{(1/2-i\sqrt{3}/2)t}$ and $y_c = e^{(1/2+i\sqrt{3}/2)t}$. Since the multiplicity of the $r = 0$ root is 3, a second and third linearly independent solution can be obtained from the first by including factors of t and t^2: $y_c = te^0 = t$ and $y_c = t^2e^0 = t^2$. By the principle of superposition, the general solution for y_c is a linear combination of these six.

$$\begin{aligned}
y_c(t) &= C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_5e^{(1/2-i\sqrt{3}/2)t} + C_6e^{(1/2+i\sqrt{3}/2)t} \\
&= C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_5e^{t/2-i\sqrt{3}t/2} + C_6e^{t/2+i\sqrt{3}t/2} \\
&= C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_5e^{t/2}e^{-i\sqrt{3}t/2} + C_6e^{t/2}e^{i\sqrt{3}t/2} \\
&= C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_5e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t\right) + C_6e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right) \\
&= C_1e^{-t} + C_2 + C_3t + C_4t^2 + e^{t/2}(C_5 + C_6)\cos\frac{\sqrt{3}}{2}t + e^{t/2}(-iC_5 + iC_6)\sin\frac{\sqrt{3}}{2}t \\
&= C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_7e^{t/2}\cos\frac{\sqrt{3}}{2}t + C_8e^{t/2}\sin\frac{\sqrt{3}}{2}t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(6)} + y_p''' = t.$$

To account for the inhomogeneous term, we would include $A + Bt$ in the trial solution, but because 1 and t already satisfy equation (1) (as well as t^2), an additional factor of t^3 is needed. Substitute the trial solution $y_p(t) = t^3(A + Bt)$ in the ODE to determine A and B.

$$[t^3(A + Bt)]^{(6)} + [t^3(A + Bt)]''' = t$$

Evaluate the derivatives.

$$(0) + (6A + 24Bt) = t$$

Simplify the left side.

$$6A + 24Bt = t$$

Match the coefficients to obtain a system of equations for A and B.

$$A = 0$$
$$24B = 1$$

Solving this system yields $A = 0$ and $B = 1/24$. As a result, the particular solution is $y_p(t) = t^3(t/24)$, and the general solution is

$$y(t) = C_1e^{-t} + C_2 + C_3t + C_4t^2 + C_7e^{t/2}\cos\frac{\sqrt{3}}{2}t + C_8e^{t/2}\sin\frac{\sqrt{3}}{2}t + \frac{t^4}{24}.$$