Problem 10

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

$$y^{(4)} + 2y'' + y = 3t + 4; \qquad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c'' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 2(r^2e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 + 2r^2 + 1 = 0$$
$$(r^2+1)^2 = 0$$
$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of t: $y_c = te^{-it}$ and $y_c = te^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1e^{-it} + C_2e^{it} + C_3te^{-it} + C_4te^{it} \\ &= C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t) + C_3t(\cos t - i\sin t) + C_4t(\cos t + i\sin t) \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + t(C_3 + C_4)\cos t + t(-iC_3 + iC_4)\sin t \\ &= C_5\cos t + C_6\sin t + C_7t\cos t + C_8t\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p'' + y_p = 3t + 4.$$

To account for the two terms on the right side, we will include $A + Bt$ in the trial solution. Substitute $y_p(t) = A + Bt$ in the ODE to determine A and B.

$$(A+Bt)^{(4)} + 2(A+Bt)'' + (A+Bt) = 3t + 4$$

Evaluate the derivatives.

$$(0) + 2(0) + A + Bt = 3t + 4$$

Simplify the left side.

$$A + Bt = 3t + 4$$

Match the coefficients to obtain a system of equations for A and B.

$$A = 4$$
$$B = 3$$

As a result, the particular solution is $y_p(t) = 4 + 3t$, and the general solution is

$$y(t) = C_5 \cos t + C_6 \sin t + C_7 t \cos t + C_8 t \sin t + 4 + 3t.$$

Differentiate it with respect to t three times.

$$y'(t) = -C_5 \sin t + C_6 \cos t + C_7 \cos t - C_7 t \sin t + C_8 \sin t + C_8 t \cos t + 3$$
$$y''(t) = -C_5 \cos t - C_6 \sin t - C_7 \sin t - C_7 \sin t - C_7 t \cos t + C_8 \cos t + C_8 \cos t - C_8 t \sin t$$
$$y'''(t) = C_5 \sin t - C_6 \cos t - C_7 \cos t - C_7 \cos t - C_7 \cos t + C_7 t \sin t - C_8 \sin t - C_8 \sin t - C_8 \sin t - C_8 t \cos t$$

Now apply the initial conditions to determine C_5, C_6, C_7, and C_8.

$$y(0) = C_5 + 4 = 0$$
$$y'(0) = C_6 + C_7 + 3 = 0$$
$$y''(0) = -C_5 + 2C_8 = 1$$
$$y'''(0) = -C_6 - 3C_7 = 1$$

Solving this system of equations yields $C_5 = -4$, $C_6 = -4$, $C_7 = 1$, and $C_8 = -3/2$. Therefore,

$$y(t) = -4 \cos t - 4 \sin t + t \cos t - \frac{3}{2} t \sin t + 4 + 3t.$$