Problem 18

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

$$y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c''' + 2y_c'' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 2(r^3e^{rt}) + 2(r^2e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 2r^3 + 2r^2 = 0$$

$$r^2(r^2 + 2r + 2) = 0$$

Use the zero product theorem.

$$\begin{aligned} r^2 = 0 \qquad &\text{or} \qquad r^2 + 2r + 2 = 0 \\ r = 0 \qquad &\text{or} \qquad r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \end{aligned}$$

$$r = \{0, -1 - i, -1 + i\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{(-1-i)t}$ and $y_c = e^{(-1+i)t}$. Since the multiplicity of the $r = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^0 = t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 + C_2t + C_3e^{(-1-i)t} + C_4e^{(-1+i)t} \\ &= C_1 + C_2t + C_3e^{-t-it} + C_4e^{-t+it} \\ &= C_1 + C_2t + C_3e^{-t}e^{-it} + C_4e^{-t}e^{it} \\ &= C_1 + C_2t + C_3e^{-t}(\cos t - i\sin t) + C_4e^{-t}(\cos t + i\sin t) \\ &= C_1 + C_2t + (C_3 + C_4)e^{-t}\cos t + (-iC_3 + iC_4)e^{-t}\sin t \\ &= C_1 + C_2t + C_5e^{-t}\cos t + C_6e^{-t}\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p''' + 2y_p'' = 3e^t + 2te^{-t} + e^{-t}\sin t.$$

There are three terms on the right side. To account for the first one, we will include Ae^t in the trial solution. To account for the second term, we will include $(B + Ct)e^{-t}$ in the trial solution. To account for the third term, we would include $e^{-t}(D\cos t + E\sin t)$ in the trial solution, but because $e^{-t}\sin t$ satisfies equation (1), an extra factor of t is needed. Therefore, the trial solution to plug in is

$$y_p(t) = Ae^t + (B + Ct)e^{-t} + te^{-t}(D\cos t + E\sin t).$$