Problem 19

Consider the nonhomogeneous nth order linear differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = g(t), \tag{i}$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces Eq. (i) to the form

$$k_0u^{(n)} + k_1u^{(n-1)} + \cdots + k_nu = b_0t^m + \cdots + b_m, \tag{ii}$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Solution

Let $g(t) = e^{\alpha t}(b_0t^m + \cdots + b_m)$. Then Eq. (i) becomes

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = e^{\alpha t}(b_0t^m + \cdots + b_m).$$

Make the substitution $y = e^{\alpha t}u(t)$.

$$a_0[e^{\alpha t}u(t)]^{(n)} + a_1[e^{\alpha t}u(t)]^{(n-1)} + \cdots + a_n[e^{\alpha t}u(t)] = e^{\alpha t}(b_0t^m + \cdots + b_m) \tag{1}$$

Start taking derivatives of $e^{\alpha t}u(t)$ to observe a pattern.

$$\begin{aligned}
[e^{\alpha t}u(t)]' &= e^{\alpha t}(\alpha u + u') \\
[e^{\alpha t}u(t)]'' &= e^{\alpha t}(\alpha^2u + 2\alpha u' + u'') \\
[e^{\alpha t}u(t)]''' &= e^{\alpha t}(\alpha^3u + 3\alpha^2u' + 3\alpha u'' + u''') \\
[e^{\alpha t}u(t)]^{(4)} &= e^{\alpha t}[\alpha^4u + 4\alpha^3u' + 6\alpha^2u'' + 4\alpha u''' + u^{(4)}] \\
&\vdots \\
[e^{\alpha t}u(t)]^{(n)} &= e^{\alpha t}[\alpha^nu + \cdots + u^{(n)}]
\end{aligned}$$

As a result, substituting these expressions into equation (1),

$$a_0e^{\alpha t}[\alpha^nu + \cdots + u^{(n)}] + a_1e^{\alpha t}[\alpha^{n-1}u + \cdots + u^{(n-1)}] + \cdots + a_n[e^{\alpha t}u(t)] = e^{\alpha t}(b_0t^m + \cdots + b_m)$$

$$a_0e^{\alpha t}u^{(n)} + \cdots + e^{\alpha t}(a_0\alpha^nu + a_1\alpha^{n-1}u + \cdots + a_{n-1}\alpha u + a_nu) = e^{\alpha t}(b_0t^m + \cdots + b_m).$$

Divide both sides by $e^{\alpha t}$ and factor u.

$$a_0u^{(n)} + \cdots + (a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n)u = e^{\alpha t}(b_0t^m + \cdots + b_m)$$

Therefore, comparing the coefficients,

$$\begin{aligned}
k_0 &= a_0 \\
k_n &= a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n.
\end{aligned}$$