Problem 22

Use the method of annihilators to find the form of a particular solution Y(t) for each of the equations in Problems 13 through 18. Do not evaluate the coefficients.

Solution

Problem 13

$$y''' - 2y'' + y' = t^3 + 2e^t$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - 2y_c'' + y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - 2(r^2e^{rt}) + re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - 2r^2 + r = 0$$

$$r(r-1)^2 = 0$$

$$r = \{0, 1\}$$

Two solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^t$. Since the multiplicity of the $r = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1 + C_2e^t + C_3te^t$$

On the other hand, the particular solution satisfies

$$y_p''' - 2y_p'' + y_p' = t^3 + 2e^t.$$

Apply the method of annihilators to determine y_p.

$$(D^3 - 2D^2 + D)y_p = t^3 + 2e^t$$

$$D(D-1)^2y_p = t^3 + 2e^t$$

The operator D^4 annihilates t^3, and the operator $D - 1$ annihilates $2e^t$. Apply the operator $D^4(D - 1)$ to both sides then.

$$D^4(D-1)D(D-1)^2y_p = D^4(D-1)(t^3 + 2e^t)$$

$$\begin{aligned} D^5(D-1)^3y_p &= D^4(D-1)(t^3+2e^t) \\ &= (D^5-D^4)(t^3+2e^t) \\ &= D^5(t^3+2e^t) - D^4(t^3+2e^t) \\ &= D^5(t^3) + D^5(2e^t) - D^4(t^3) - D^4(2e^t) \\ &= 0 + 2e^t - 0 - 2e^t \\ &= 0 \end{aligned}$$

Expand the left side.

$$(D^8 - 3D^7 + 3D^6 - D^5)y_p = 0$$

$$y_p^{(8)} - 3y_p^{(7)} + 3y_p^{(6)} - y_p^{(5)} = 0 \tag{2}$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad \rightarrow \quad y_p^{(5)} = s^5e^{st} \quad \rightarrow \quad y_p^{(6)} = s^6e^{st} \quad \rightarrow \quad y_p^{(7)} = s^3e^{st} \quad \rightarrow \quad y_p^{(8)} = s^8e^{st}$$

Substitute these expressions into the ODE.

$$s^8e^{st} - 3(s^7e^{st}) + 3(s^6e^{st}) - s^5e^{st} = 0$$

Divide both sides by e^{st}.

$$s^8 - 3s^7 + 3s^6 - s^5 = 0$$

$$s^5(s-1)^3 = 0$$

$$s = \{0, 1\}$$

Two solutions to equation (2) are $y_p = e^0 = 1$ and $y_p = e^t$. The multiplicity of the $s = 0$ root is 5, so a second, third, fourth, and fifth solution can be obtained from the first by including factors of t, t^2, t^3, and t^4: $y_p = te^0 = t$ and $y_p = t^2e^0 = t^2$ and $y_p = t^3e^0 = t^3$ and $y_p = t^4e^0 = t^4$. The multiplicity of the $s = 1$ root is 3, so a second and third solution can be obtained from the first by including factors of t and t^2: $y_p = te^t$ and $y_p = t^2e^t$. By the principle of superposition, the general solution for y_p is a linear combination of these eight.

$$y_p(t) = C_4 + C_5t + C_6t^2 + C_7t^3 + C_8t^4 + C_9e^t + C_{10}te^t + C_{11}t^2e^t$$

Comparing this to $y_c(t)$, we set $C_4 = 0$, $C_9 = 0$, and $C_{10} = 0$ so that no terms are shared in common.

$$y_p(t) = C_5t + C_6t^2 + C_7t^3 + C_8t^4 + C_{11}t^2e^t$$

Problem 14

$$y''' - y' = te^{-t} + 2\cos t$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r = 0$$

$$r(r+1)(r-1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = te^{-t} + 2\cos t.$$

Apply the method of annihilators to determine y_p.

$$(D^3 - D)y_p = te^{-t} + 2\cos t$$

$$D(D-1)(D+1)y_p = te^{-t} + 2\cos t$$

The operator $(D+1)^2$ annihilates te^{-t}, and the operator D^2+1 annihilates $2\cos t$. Apply the operator $(D+1)^2(D^2+1)$ to both sides then.

$$(D+1)^2(D^2+1)D(D-1)(D+1)y_p = (D+1)^2(D^2+1)(te^{-t} + 2\cos t)$$

$$\begin{aligned}
D(D^2+1)(D-1)(D+1)^3 y_p &= (D^2+2D+1)(D^2+1)(te^{-t}+2\cos t) \\
&= (D^4+2D^3+2D^2+2D+1)(te^{-t}+2\cos t) \\
&= D^4(te^{-t}+2\cos t)+2D^3(te^{-t}+2\cos t)+2D^2(te^{-t}+2\cos t) \\
&\quad\ + 2D(te^{-t}+2\cos t)+(te^{-t}+2\cos t) \\
&= D^3(e^{-t}-te^{-t}-2\sin t)+2D^2(e^{-t}-te^{-t}-2\sin t) \\
&\quad\ + 2D(e^{-t}-te^{-t}-2\sin t)+2(e^{-t}-te^{-t}-2\sin t)+(te^{-t}+2\cos t) \\
&= D^2(-e^{-t}-e^{-t}+te^{-t}-2\cos t)+2D(-e^{-t}-e^{-t}+te^{-t}-2\cos t) \\
&\quad\ + 2(-e^{-t}-e^{-t}+te^{-t}-2\cos t)+2(e^{-t}-te^{-t}-2\sin t) \\
&\quad\quad + (te^{-t}+2\cos t) \\
&= D(e^{-t}+e^{-t}+e^{-t}-te^{-t}+2\sin t)+2(e^{-t}+e^{-t}+e^{-t}-te^{-t}+2\sin t) \\
&\quad\ + 2(-e^{-t}-e^{-t}+te^{-t}-2\cos t)+2(e^{-t}-te^{-t}-2\sin t) \\
&\quad\quad + (te^{-t}+2\cos t) \\
&= (-e^{-t}-e^{-t}-e^{-t}-e^{-t}+te^{-t}+2\cos t)+2(e^{-t}+e^{-t}+e^{-t}-te^{-t}+2\sin t) \\
&\quad\ + 2(-e^{-t}-e^{-t}+te^{-t}-2\cos t)+2(e^{-t}-te^{-t}-2\sin t) \\
&\quad\quad + (te^{-t}+2\cos t) \\
&= 0
\end{aligned}$$

Expand the left side.

$$(D^7+2D^6+D^5-D^3-2D^2-D)y_p = 0$$

$$y_p^{(7)}+2y_p^{(6)}+y_p^{(5)}-y_p'''-2y_p''-y_p' = 0 \tag{2}$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad\rightarrow\quad y_p' = se^{st} \quad\rightarrow\quad y_p^{(5)} = s^5e^{st} \quad\rightarrow\quad y_p^{(7)} = s^7e^{st}$$

Substitute these expressions into the ODE.

$$s^7e^{st}+2(s^6e^{st})+s^5e^{st}-s^3e^{st}-2(s^2e^{st})-se^{st} = 0$$

Divide both sides by e^{st}.

$$\begin{gathered}
s^7+2s^6+s^5-s^3-2s^2-s = 0 \\
s(s^2+1)(s+1)^3(s-1) = 0 \\
s = \{-1, 0, 1, -i, i\}
\end{gathered}$$

Five solutions to equation (2) are $y_p = e^{-t}$ and $y_p = e^0 = 1$ and $y_p = e^t$ and $y_p = e^{-it}$ and $y_p = e^{it}$. The multiplicity of the $s = -1$ root is 3, so a second and third solution can be obtained from the first by including factors of t and t^2: $y_p = te^{-t}$ and $y_p = t^2e^{-t}$. By the principle of superposition, the general solution for y_p is a linear combination of these seven.

$$\begin{aligned}
y_p(t) &= C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4+C_5e^t+C_6e^{-it}+C_7e^{it} \\
&= C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4+C_5e^t+C_6(\cos t-i\sin t)+C_7(\cos t+i\sin t) \\
&= C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4+C_5e^t+(C_6+C_7)\cos t+(-iC_6+iC_7)\sin t \\
&= C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4+C_5e^t+C_8\cos t+C_9\sin t
\end{aligned}$$

Comparing this to $y_c(t)$, we set $C_1 = 0$, $C_4 = 0$, and $C_5 = 0$ so that no terms are shared in common.

$$y_p(t) = C_2te^{-t}+C_3t^2e^{-t}+C_8\cos t+C_9\sin t$$

Problem 15

$$y^{(4)} - 2y'' + y = e^t + \sin t$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - 2y_c'' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - 2(r^2e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - 2r^2 + 1 = 0$$
$$(r^2 - 1)^2 = 0$$
$$r = \{-1, 1\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^t$. Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of t: $y_c = te^{-t}$ and $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$y_c(t) = C_1e^{-t} + C_2e^t + C_3te^{-t} + C_4te^t$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - 2y_p'' + y_p = e^t + \sin t.$$

Apply the method of annihilators to determine y_p.

$$(D^4 - 2D^2 + 1)y_p = e^t + \sin t$$
$$(D - 1)^2(D + 1)^2y_p = e^t + \sin t$$

The operator $D - 1$ annihilates e^t, and the operator $D^2 + 1$ annihilates $\sin t$. Apply the operator $(D - 1)(D^2 + 1)$ to both sides then.

$$(D - 1)(D^2 + 1)(D - 1)^2(D + 1)^2y_p = (D - 1)(D^2 + 1)(e^t + \sin t)$$

$$\begin{aligned}
(D^2 + 1)(D - 1)^3(D + 1)^2y_p &= (D - 1)(D^2 + 1)(e^t + \sin t) \\
&= (D^3 - D^2 + D - 1)(e^t + \sin t) \\
&= D^3(e^t + \sin t) - D^2(e^t + \sin t) + D(e^t + \sin t) - (e^t + \sin t) \\
&= D^2(e^t + \cos t) - D(e^t + \cos t) + (e^t + \cos t) - (e^t + \sin t) \\
&= D(e^t - \sin t) - (e^t - \sin t) + (e^t + \cos t) - (e^t + \sin t) \\
&= (e^t - \cos t) - (e^t - \sin t) + (e^t + \cos t) - (e^t + \sin t) \\
&= 0
\end{aligned}$$

Expand the left side.

$$(D^7 - D^6 - D^5 + D^4 - D^3 + D^2 + D - 1)y_p = 0$$

$$y_p^{(7)} - y_p^{(6)} - y_p^{(5)} + y_p^{(4)} - y_p''' + y_p'' + y_p' - y_p = 0 \tag{2}$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad \rightarrow \quad y_p' = se^{st} \quad \rightarrow \quad y_p^{(5)} = s^5 e^{st} \quad \rightarrow \quad y_p^{(7)} = s^7 e^{st}$$

Substitute these expressions into the ODE.

$$s^7 e^{st} - s^6 e^{st} - s^5 e^{st} + s^4 e^{st} - s^3 e^{st} + s^2 e^{st} + se^{st} - e^{st} = 0$$

Divide both sides by e^{st}.

$$s^7 - s^6 - s^5 + s^4 - s^3 + s^2 + s - 1 = 0$$

$$(s^2 + 1)(s + 1)^2 (s - 1)^3 = 0$$

$$s = \{-1, 1, -i, i\}$$

Four solutions to equation (2) are then $y_p = e^{-t}$ and $y_p = e^t$ and $y_p = e^{-it}$ and $y_p = e^{it}$. Since the multiplicity of the $s = -1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^{-t}$. Since the multiplicity of the $s = 1$ root is 3, a second and third linearly independent solution can be obtained from the first by including factors of t and t^2: $y_p = te^t$ and $y_p = t^2 e^t$. By the principle of superposition, the general solution for y_p is a linear combination of these seven.

$$\begin{aligned}
y_p(t) &= C_1 e^{-t} + C_2 te^{-t} + C_3 e^t + C_4 te^t + C_5 t^2 e^t + C_6 e^{-it} + C_7 e^{it} \\
&= C_1 e^{-t} + C_2 te^{-t} + C_3 e^t + C_4 te^t + C_5 t^2 e^t + C_6(\cos t - i \sin t) + C_7(\cos t + i \sin t) \\
&= C_1 e^{-t} + C_2 te^{-t} + C_3 e^t + C_4 te^t + C_5 t^2 e^t + (C_6 + C_7) \cos t + (-iC_6 + iC_7) \sin t \\
&= C_1 e^{-t} + C_2 te^{-t} + C_3 e^t + C_4 te^t + C_5 t^2 e^t + C_8 \cos t + C_9 \sin t
\end{aligned}$$

Comparing this to $y_c(t)$, we set $C_1 = 0$, $C_2 = 0$, $C_3 = 0$, and $C_4 = 0$ so that no terms are shared in common.

$$y_p(t) = C_5 t^2 e^t + C_8 \cos t + C_9 \sin t$$

Problem 16

$$y^{(4)} + 4y'' = \sin 2t + te^t + 4$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 4y_c'' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 4(r^2e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 4r^2 = 0$$

$$r^2(r^2 + 4) = 0$$

$$r = \{0, -2i, 2i\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{-2it}$ and $y_c = e^{2it}$. Since the multiplicity of the $r = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^0 = t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 + C_2t + C_3e^{-2it} + C_4e^{2it} \\ &= C_1 + C_2t + C_3(\cos 2t - i\sin 2t) + C_4(\cos 2t + i\sin 2t) \\ &= C_1 + C_2t + (C_3 + C_4)\cos 2t + (-iC_3 + iC_4)\sin 2t \\ &= C_1 + C_2t + C_5\cos 2t + C_6\sin 2t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 4y_p'' = \sin 2t + te^t + 4.$$

Apply the method of annihilators to determine y_p.

$$(D^4 + 4D^2)y_p = \sin 2t + te^t + 4$$

$$D^2(D^2 + 4)y_p = \sin 2t + te^t + 4$$

The operator $D^2 + 4$ annihilates $\sin 2t$, the operator $(D - 1)^2$ annihilates te^t, and the operator D annihilates 4. Apply the operator $D(D^2 + 4)(D - 1)^2$ to both sides then.

$$D(D^2 + 4)(D - 1)^2D^2(D^2 + 4)y_p = D(D^2 + 4)(D - 1)^2(\sin 2t + te^t + 4)$$

$$\begin{aligned}
D^3(D^2+4)^2(D-1)^2 y_p &= (D^5 - 2D^4 + 5D^3 - 8D^2 + 4D)(\sin 2t + te^t + 4) \\
&= D^5(\sin 2t + te^t + 4) - 2D^4(\sin 2t + te^t + 4) + 5D^3(\sin 2t + te^t + 4) \\
&\qquad - 8D^2(\sin 2t + te^t + 4) + 4D(\sin 2t + te^t + 4) \\
&= D^4(2\cos 2t + e^t + te^t) - 2D^3(2\cos 2t + e^t + te^t) + 5D^2(2\cos 2t + e^t + te^t) \\
&\qquad - 8D(2\cos 2t + e^t + te^t) + 4(2\cos 2t + e^t + te^t) \\
&= D^3(-4\sin 2t + e^t + e^t + te^t) - 2D^2(-4\sin 2t + e^t + e^t + te^t) \\
&\qquad + 5D(-4\sin 2t + e^t + e^t + te^t) - 8(-4\sin 2t + e^t + e^t + te^t) + 4(2\cos 2t + e^t + te^t) \\
&= D^2(-8\cos 2t + e^t + e^t + e^t + te^t) - 2D(-8\cos 2t + e^t + e^t + e^t + te^t) \\
&\qquad + 5(-8\cos 2t + e^t + e^t + e^t + te^t) - 8(-4\sin 2t + e^t + e^t + te^t) \\
&\qquad\qquad + 4(2\cos 2t + e^t + te^t) \\
&= D(16\sin 2t + e^t + e^t + e^t + e^t + te^t) - 2(16\sin 2t + e^t + e^t + e^t + e^t + te^t) \\
&\qquad + 5(-8\cos 2t + e^t + e^t + e^t + te^t) - 8(-4\sin 2t + e^t + e^t + te^t) \\
&\qquad\qquad + 4(2\cos 2t + e^t + te^t) \\
&= (32\cos 2t + e^t + e^t + e^t + e^t + e^t + te^t) - 2(16\sin 2t + e^t + e^t + e^t + e^t + te^t) \\
&\qquad + 5(-8\cos 2t + e^t + e^t + e^t + te^t) - 8(-4\sin 2t + e^t + e^t + te^t) \\
&\qquad\qquad + 4(2\cos 2t + e^t + te^t) \\
&= 0
\end{aligned}$$

Expand the left side.

$$(D^9 - 2D^8 + 9D^7 - 16D^6 + 24D^5 - 32D^4 + 16D^3)y_p = 0$$

$$y_p^{(9)} - 2y_p^{(8)} + 9y_p^{(7)} - 16y_p^{(6)} + 24y_p^{(5)} - 32y_p^{(4)} + 16y_p''' = 0 \qquad (2)$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad \to \quad y_p' = se^{st} \quad \to \quad y_p^{(5)} = s^5 e^{st} \quad \to \quad y_p^{(9)} = s^9 e^{st}$$

Substitute these expressions into the ODE.

$$s^9 e^{st} - 2(s^8 e^{st}) + 9(s^7 e^{st}) - 16(s^6 e^{st}) + 24(s^5 e^{st}) - 32(s^4 e^{st}) + 16(s^3 e^{st}) = 0$$

Divide both sides by e^{st}.

$$s^9 - 2s^8 + 9s^7 - 16s^6 + 24s^5 - 32s^4 + 16s^3 = 0$$

$$s^3(s-1)^2(s^2+4)^2 = 0$$

$$s = \{0, 1, -2i, 2i\}$$

Four solutions to equation (2) are then $y_p = e^0 = 1$ and $y_p = e^t$ and $y_p = e^{-2it}$ and $y_p = e^{2it}$. Since the multiplicity of the $s = 0$ root is 3, a second and third linearly independent solution can be obtained from the first by including factors of t and t^2: $y_p = te^0 = t$ and $y_p = t^2 e^0 = t^2$. Since the multiplicity of the $s = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^t$. Since the multiplicity of the $s = -2i$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^{-2it}$. Since the multiplicity of the $s = 2i$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^{2it}$. By the principle of superposition, the general solution for y_p is a linear combination of these nine.

$$\begin{aligned} y_p(t) &= C_7 + C_8t + C_9t^2 + C_{10}e^t + C_{11}te^t + C_{12}e^{-2it} + C_{13}te^{-2it} + C_{14}e^{2it} + C_{15}te^{2it} \\ &= C_7 + C_8t + C_9t^2 + C_{10}e^t + C_{11}te^t + C_{12}(\cos 2t - i\sin 2t) + C_{13}t(\cos 2t - i\sin 2t) \\ &\quad\quad + C_{14}(\cos 2t + i\sin 2t) + C_{15}t(\cos 2t + i\sin 2t) \\ &= C_7 + C_8t + C_9t^2 + C_{10}e^t + C_{11}te^t + (C_{12} + C_{14})\cos 2t + (-iC_{12} + iC_{14})\sin 2t \\ &\quad\quad + (C_{13} + C_{15})t\cos 2t + (-iC_{13} + iC_{15})t\sin 2t \\ &= C_7 + C_8t + C_9t^2 + C_{10}e^t + C_{11}te^t + C_{16}\cos 2t + C_{17}\sin 2t + C_{18}t\cos 2t + C_{19}t\sin 2t \end{aligned}$$

Comparing this to $y_c(t)$, we set $C_7 = 0$, $C_8 = 0$, $C_{16} = 0$, and $C_{17} = 0$ so that no terms are shared in common.

$$y_p(t) = C_9t^2 + C_{10}e^t + C_{11}te^t + C_{18}t\cos 2t + C_{19}t\sin 2t$$

Problem 17

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - y_c''' - y_c'' + y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} - r^3e^{rt} - r^2e^{rt} + re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - r^3 - r^2 + r = 0$$

$$r(r+1)(r-1)^2 = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. Since the multiplicity of the $r = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t + C_4te^t$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - y_p''' - y_p'' + y_p' = t^2 + 4 + t\sin t.$$

Apply the method of annihilators to determine y_p.

$$(D^4 - D^3 - D^2 + D)y_p = t^2 + 4 + t\sin t$$

$$D(D+1)(D-1)^2y_p = t^2 + 4 + t\sin t$$

The operator D^3 annihilates $t^2 + 4$, and the operator $(D^2+1)^2$ annihilates $t\sin t$. Apply the operator $D^3(D^2+1)^2$ to both sides then.

$$D^3(D^2+1)^2D(D+1)(D-1)^2y_p = D^3(D^2+1)^2(t^2 + 4 + t\sin t)$$

$$\begin{aligned}
D^4(D^2+1)^2(D+1)(D-1)^2 y_p &= (D^7+2D^5+D^3)(t^2+4+t\sin t) \\
&= D^7(t^2+4+t\sin t)+2D^5(t^2+4+t\sin t)+D^3(t^2+4+t\sin t) \\
&= D^6(2t+\sin t+t\cos t)+2D^4(2t+\sin t+t\cos t)+D^2(2t+\sin t+t\cos t) \\
&= D^5(2+\cos t+\cos t-t\sin t)+2D^3(2+\cos t+\cos t-t\sin t) \\
&\qquad + D(2+\cos t+\cos t-t\sin t) \\
&= D^4(-\sin t-\sin t-\sin t-t\cos t)+2D^2(-\sin t-\sin t-\sin t-t\cos t) \\
&\qquad + (-\sin t-\sin t-\sin t-t\cos t) \\
&= D^3(-\cos t-\cos t-\cos t-\cos t+t\sin t) \\
&\qquad + 2D(-\cos t-\cos t-\cos t-\cos t+t\sin t) \\
&\qquad\qquad + (-\sin t-\sin t-\sin t-t\cos t) \\
&= D^2(\sin t+\sin t+\sin t+\sin t+\sin t+t\cos t) \\
&\qquad + 2(\sin t+\sin t+\sin t+\sin t+\sin t+t\cos t) \\
&\qquad\qquad + (-\sin t-\sin t-\sin t-t\cos t) \\
&= D(\cos t+\cos t+\cos t+\cos t+\cos t+\cos t-t\sin t) \\
&\qquad + 2(\sin t+\sin t+\sin t+\sin t+\sin t+t\cos t) \\
&\qquad\qquad + (-\sin t-\sin t-\sin t-t\cos t) \\
&= (-\sin t-\sin t-\sin t-\sin t-\sin t-\sin t-\sin t-t\cos t) \\
&\qquad + 2(\sin t+\sin t+\sin t+\sin t+\sin t+t\cos t) \\
&\qquad\qquad + (-\sin t-\sin t-\sin t-t\cos t) \\
&= 0
\end{aligned}$$

Expand the left side.

$$(D^{11}-D^{10}+D^9-D^8-D^7+D^6-D^5+D^4)y_p = 0$$

$$y_p^{(11)}-y_p^{(10)}+y_p^{(9)}-y_p^{(8)}-y_p^{(7)}+y_p^{(6)}-y_p^{(5)}+y_p^{(4)} = 0 \tag{2}$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad\to\quad y_p' = se^{st} \quad\to\quad y_p^{(5)} = s^5e^{st} \quad\to\quad y_p^{(9)} = s^9e^{st} \quad\to\quad y_p^{(11)} = s^{11}e^{st}$$

Substitute these expressions into the ODE.

$$s^{11}e^{st}-s^{10}e^{st}+s^9e^{st}-s^8e^{st}-s^7e^{st}+s^6e^{st}-s^5e^{st}+s^4e^{st} = 0$$

Divide both sides by e^{st}.

$$s^{11}-s^{10}+s^9-s^8-s^7+s^6-s^5+s^4 = 0$$

$$s^4(s+1)(s-1)^2(s^2+1)^2 = 0$$

$$s = \{-1, 0, 1, -i, i\}$$

Five solutions to equation (2) are then $y_p = e^{-t}$ and $y_p = e^0 = 1$ and $y_p = e^t$ and $y_p = e^{-it}$ and $y_p = e^{it}$. Since the multiplicity of the $s = 0$ root is 4, a second, third, and fourth linearly independent solution can be obtained from the first by including factors of t, t^2, and t^3: $y_p = te^0 = t$ and $y_p = t^2e^0 = t^2$ and $y_p = t^3e^0 = t^3$. Since the multiplicity of the $s = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^t$. Since the multiplicity of the $s = -i$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^{-it}$. Since the multiplicity of the $s = i$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_p = te^{it}$. By the principle of superposition, the general solution for y_p is a linear combination of these eleven.

$$\begin{aligned}
y_p(t) &= C_5e^{-t} + C_6 + C_7t + C_8t^2 + C_9t^3 + C_{10}e^t + C_{11}te^t + C_{12}e^{-it} + C_{13}te^{-it} + C_{14}e^{it} + C_{15}te^{it} \\
&= C_5e^{-t} + C_6 + C_7t + C_8t^2 + C_9t^3 + C_{10}e^t + C_{11}te^t + C_{12}(\cos t - i\sin t) + C_{13}t(\cos t - i\sin t) \\
&\qquad + C_{14}(\cos t + i\sin t) + C_{15}t(\cos t + i\sin t) \\
&= C_5e^{-t} + C_6 + C_7t + C_8t^2 + C_9t^3 + C_{10}e^t + C_{11}te^t + (C_{12} + C_{14})\cos t + (-iC_{12} + iC_{14})\sin t \\
&\qquad + (C_{13} + C_{15})t\cos t + (-iC_{13} + iC_{15})t\sin t \\
&= C_5e^{-t} + C_6 + C_7t + C_8t^2 + C_9t^3 + C_{10}e^t + C_{11}te^t + C_{16}\cos t + C_{17}\sin t + C_{18}t\cos t + C_{19}t\sin t
\end{aligned}$$

Comparing this to $y_c(t)$, we set $C_5 = 0$, $C_6 = 0$, $C_{10} = 0$, and $C_{11} = 0$ so that no terms are shared in common.

$$y_p(t) = C_7t + C_8t^2 + C_9t^3 + C_{16}\cos t + C_{17}\sin t + C_{18}t\cos t + C_{19}t\sin t$$

Problem 18

$$y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$$

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c''' + 2y_c'' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} + 2(r^3 e^{rt}) + 2(r^2 e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 + 2r^3 + 2r^2 = 0$$

$$r^2(r^2 + 2r + 2) = 0$$

Use the zero product theorem.

$$\begin{aligned} r^2 = 0 \quad &\text{or} \quad r^2 + 2r + 2 = 0 \\ r = 0 \quad &\text{or} \quad r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \end{aligned}$$

$$r = \{0, -1 - i, -1 + i\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{(-1-i)t}$ and $y_c = e^{(-1+i)t}$. Since the multiplicity of the $r = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^0 = t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 + C_2 t + C_3 e^{(-1-i)t} + C_4 e^{(-1+i)t} \\ &= C_1 + C_2 t + C_3 e^{-t-it} + C_4 e^{-t+it} \\ &= C_1 + C_2 t + C_3 e^{-t} e^{-it} + C_4 e^{-t} e^{it} \\ &= C_1 + C_2 t + C_3 e^{-t}(\cos t - i \sin t) + C_4 e^{-t}(\cos t + i \sin t) \\ &= C_1 + C_2 t + (C_3 + C_4) e^{-t} \cos t + (-iC_3 + iC_4) e^{-t} \sin t \\ &= C_1 + C_2 t + C_5 e^{-t} \cos t + C_6 e^{-t} \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p''' + 2y_p'' = 3e^t + 2te^{-t} + e^{-t}\sin t.$$

Apply the method of annihilators to determine y_p.

$$(D^4 + 2D^3 + 2D^2)y_p = 3e^t + 2te^{-t} + e^{-t}\sin t$$

$$D^2(D^2+2D+2)y_p = 3e^t + 2te^{-t} + e^{-t}\sin t$$

The operator $D-1$ annihilates $3e^t$, the operator $(D+1)^2$ annihilates $2te^{-t}$, and the operator D^2+2D+2 annihilates $e^{-t}\sin t$. Apply the operator $(D-1)(D+1)^2(D^2+2D+2)$ to both sides then.

$$(D-1)(D+1)^2(D^2+2D+2)D^2(D^2+2D+2)y_p = (D-1)(D+1)^2(D^2+2D+2)(3e^t + 2te^{-t} + e^{-t}\sin t)$$

$$\begin{aligned}
D^2(D^2+2D+2)^2(D-1)(D+1)^2y_p &= (D^5+3D^4+3D^3-D^2-4D-2)(3e^t+2te^{-t}+e^{-t}\sin t) \\
&= D^5(3e^t+2te^{-t}+e^{-t}\sin t) + 3D^4(3e^t+2te^{-t}+e^{-t}\sin t) \\
&\quad + 3D^3(3e^t+2te^{-t}+e^{-t}\sin t) - D^2(3e^t+2te^{-t}+e^{-t}\sin t) \\
&\qquad - 4D(3e^t+2te^{-t}+e^{-t}\sin t) - 2(3e^t+2te^{-t}+e^{-t}\sin t) \\
&= 0
\end{aligned}$$

Expand the left side.

$$(D^9+5D^8+11D^7+11D^6-12D^4-12D^3-4D^2)y_p = 0$$

$$y_p^{(9)} + 5y_p^{(8)} + 11y_p^{(7)} + 11y_p^{(6)} - 12y_p^{(4)} - 12y_p''' - 4y_p'' = 0 \tag{2}$$

Since each term on the left has constant coefficients, the solution is of the form $y_p = e^{st}$.

$$y_p = e^{st} \quad\rightarrow\quad y_p' = se^{st} \quad\rightarrow\quad y_p^{(5)} = s^5e^{st} \quad\rightarrow\quad y_p^{(9)} = s^9e^{st}$$

Substitute these expressions into the ODE.

$$s^9e^{st} + 5(s^8e^{st}) + 11(s^7e^{st}) + 11(s^6e^{st}) - 12(s^4e^{st}) - 12(s^3e^{st}) - 4(s^2e^{st}) = 0$$

Divide both sides by e^{st}.

$$s^9+5s^8+11s^7+11s^6-12s^4-12s^3-4s^2 = 0$$

$$s^2(s+1)^2(s-1)(s^2+2s+2)^2 = 0$$

$$s = \{-1, 0, 1, -1-i, -1+i\}$$

Five solutions to equation (2) are then $y_p = e^{-t}$ and $y_p = e^0 = 1$ and $y_p = e^t$ and $y_p = e^{(-1-i)t}$ and $y_p = e^{(-1+i)t}$. Since every root except $s=1$ is of multiplicity 2, a second linearly independent solution can be obtained from each of them by including a factor of t: $y_p = te^0 = t$ and $y_p = te^{-t}$ and $y_p = te^{(-1-i)t}$ and $y_p = te^{(-1+i)t}$. By the principle of superposition, the general solution is a linear combination of these nine.

$$\begin{aligned}
y_p(t) &= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + C_{12}e^{(-1-i)t} + C_{13}te^{(-1-i)t} + C_{14}e^{(-1+i)t} + C_{15}te^{(-1+i)t} \\
&= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + C_{12}e^{-t-it} + C_{13}te^{-t-it} + C_{14}e^{-t+it} + C_{15}te^{-t+it} \\
&= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + C_{12}e^{-t}e^{-it} + C_{13}te^{-t}e^{-it} + C_{14}e^{-t}e^{it} + C_{15}te^{-t}e^{it} \\
&= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + C_{12}e^{-t}(\cos t - i\sin t) + C_{13}te^{-t}(\cos t - i\sin t) \\
&\quad + C_{14}e^{-t}(\cos t + i\sin t) + C_{15}te^{-t}(\cos t + i\sin t) \\
&= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + (C_{12}+C_{14})e^{-t}\cos t + (-iC_{12}+iC_{14})e^{-t}\sin t \\
&\quad + (C_{13}+C_{15})te^{-t}\cos t + (-iC_{13}+iC_{15})te^{-t}\sin t \\
&= C_7e^{-t} + C_8te^{-t} + C_9 + C_{10}t + C_{11}e^t + C_{16}e^{-t}\cos t + C_{17}e^{-t}\sin t + C_{18}te^{-t}\cos t + C_{19}te^{-t}\sin t
\end{aligned}$$

Comparing this to $y_c(t)$, we set $C_9 = 0$, $C_{10} = 0$, $C_{16} = 0$, and $C_{17} = 0$ so that no terms are shared in common.

$$y_p(t) = C_7e^{-t} + C_8te^{-t} + C_{11}e^t + C_{18}te^{-t}\cos t + C_{19}te^{-t}\sin t$$