Problem 11

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

$$y''' - 3y'' + 2y' = t + e^t; \qquad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - 3y_c'' + 2y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - 3(r^2e^{rt}) + 2(re^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 3r^2 + 2r = 0$$

$$r(r-2)(r-1) = 0$$

$$r = \{0, 1, 2\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^t$ and $y_c = e^{2t}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1 + C_2e^t + C_3e^{2t}$$

On the other hand, the particular solution satisfies

$$y_p''' - 3y_p'' + 2y_p' = t + e^t.$$

There are two terms on the right side. To account for the first one, we would include $A + Bt$ in the trial solution, but because a constant satisfies equation (1), an extra factor of t is needed. To account for the second term, we would include Ce^t, but because e^t satisfies equation (1), an extra factor of t is needed here as well. Substitute $y_p(t) = t(A + Bt) + Cte^t$ in the ODE to determine A, B, and C.

$$[t(A + Bt) + Cte^t]''' - 3[t(A + Bt) + Cte^t]'' + 2[t(A + Bt) + Cte^t]' = t + e^t$$

Evaluate the derivatives.

$$[C(t+3)e^t] - 3[2B + C(t+2)e^t] + 2[A + 2Bt + C(t+1)e^t] = t + e^t$$

Simplify the left side.

$$2A - 6B + 4Bt - Ce^t = t + e^t$$

Match the coefficients to obtain a system of equations for A and B and C.

$$\begin{aligned} 2A - 6B &= 0 \\ 4B &= 1 \\ -C &= 1 \end{aligned}$$

Solving this system yields $A = 3/4$, $B = 1/4$, and $C = -1$. As a result, the particular solution is $y_p(t) = t(3/4 + t/4) - te^t$, and the general solution is

$$y(t) = C_1 + C_2e^t + C_3e^{2t} + \frac{3}{4}t + \frac{1}{4}t^2 - te^t.$$

Differentiate it with respect to t twice.

$$\begin{aligned} y'(t) &= C_2e^t + 2C_3e^{2t} + \frac{3}{4} + \frac{1}{2}t - e^t - te^t \\ y''(t) &= C_2e^t + 4C_3e^{2t} + \frac{1}{2} - e^t - e^t - te^t \end{aligned}$$

Now apply the initial conditions to determine C_1, C_2, and C_3.

$$\begin{aligned} y(0) &= C_1 + C_2 + C_3 = 1 \\ y'(0) &= C_2 + 2C_3 + \frac{3}{4} - 1 = -\frac{1}{4} \\ y''(0) &= C_2 + 4C_3 + \frac{1}{2} - 2 = -\frac{3}{2} \end{aligned}$$

Solving this system of equations yields $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$. Therefore,

$$y(t) = 1 + \frac{3}{4}t + \frac{1}{4}t^2 - te^t.$$