Problem 13

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

$$y''' - 2y'' + y' = t^3 + 2e^t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - 2y_c'' + y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - 2(r^2 e^{rt}) + re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - 2r^2 + r = 0$$

$$r(r-1)^2 = 0$$

$$r = \{0, 1\}$$

Two solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^t$. Since the multiplicity of the $r = 1$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of t: $y_c = te^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1 + C_2 e^t + C_3 te^t$$

On the other hand, the particular solution satisfies

$$y_p''' - 2y_p'' + y_p' = t^3 + 2e^t.$$

There are two terms on the right side. To account for the first one, we would include $A + Bt + Ct^2 + Dt^3$ in the trial solution, but because a constant satisfies equation (1), an extra factor of t is needed. To account for the second term, we would include Ee^t, but because e^t and te^t satisfy equation (1), an extra factor of t^2 is needed. Therefore, the trial solution to plug in is

$$y_p(t) = t(A + Bt + Ct^2 + Dt^3) + Et^2 e^t.$$