Problem 20

Show that linear differential operators with constant coefficients obey the commutative law. That is, show that

$$(D-a)(D-b)f = (D-b)(D-a)f$$

for any twice-differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

Solution

Evaluate the left side.

$$\begin{aligned}
(D-a)(D-b)f &= (D-a)(Df-bf) \\
&= (D-a)(f'-bf) \\
&= D(f'-bf) - a(f'-bf) \\
&= Df' - D(bf) - af' + abf \\
&= f'' - bf' - af' + abf \\
&= f'' - (b+a)f' + abf
\end{aligned}$$

Now evaluate the right side.

$$\begin{aligned}
(D-b)(D-a)f &= (D-b)(Df-af) \\
&= (D-b)(f'-af) \\
&= D(f'-af) - b(f'-af) \\
&= Df' - D(af) - bf' + baf \\
&= f'' - af' - bf' + baf \\
&= f'' - (a+b)f' + baf \\
&= f'' - (b+a)f' + abf
\end{aligned}$$

Therefore,

$$(D-a)(D-b)f = (D-b)(D-a)f.$$