Problem 4

In each of Problems 1 through 8, determine the general solution of the given differential equation.

$$y''' - y' = 2\sin t$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2e^{rt} \quad \to \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = 2\sin t.$$

To account for the inhomogeneous term, we will include $A\cos t$ in the trial solution since there are only odd derivatives on the left side. Substitute $y_p(t) = A\cos t$ in the ODE to determine A.

$$(A\cos t)''' - (A\cos t)' = 2\sin t$$

Evaluate the derivatives.

$$(A\sin t) - (-A\sin t) = 2\sin t$$

Simplify the left side.

$$2A\sin t = 2\sin t$$

Match the coefficients.

$$2A = 2$$

Solving this yields $A = 1$. As a result, the particular solution is $y_p(t) = \cos t$, and the general solution is

$$y(t) = C_1e^{-t} + C_2 + C_3e^t + \cos t.$$