Problem 9

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

$$y''' + 4y' = t; \qquad y(0) = y'(0) = 0, \quad y''(0) = 1$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' + 4y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} + 4(re^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 + 4r = 0$$

$$r(r^2 + 4) = 0$$

$$r = \{0, -2i, 2i\}$$

Two solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1 + C_2e^{-2it} + C_3e^{2it} \\ &= C_1 + C_2(\cos 2t - i\sin 2t) + C_3(\cos 2t + i\sin 2t) \\ &= C_1 + (C_2 + C_3)\cos 2t + (-iC_2 + iC_3)\sin 2t \\ &= C_1 + C_4\cos 2t + C_5\sin 2t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' + 4y_p' = t.$$

To account for the inhomogeneous term, we would include $A + Bt$ in the trial solution, but because a constant satisfies equation (1), an extra factor of t is needed. Substitute $y_p(t) = t(A + Bt)$ in the ODE to determine A and B.

$$[t(A + Bt)]''' + 4[t(A + Bt)]' = t$$

Evaluate the derivatives.

$$(0) + 4(A + 2Bt) = t$$

Simplify the left side.

$$4A + 8Bt = t$$

Match the coefficients to obtain a system of equations for A and B.

$$4A = 0$$
$$8B = 1$$

Solving this system yields $A = 0$ and $B = 1/8$. As a result, the particular solution is $y_p(t) = t(t/8)$, and the general solution is

$$y(t) = C_1 + C_4 \cos 2t + C_5 \sin 2t + \frac{t^2}{8}.$$

Differentiate it with respect to t twice.

$$y'(t) = -2C_4 \sin 2t + 2C_5 \cos 2t + \frac{t}{4}$$
$$y''(t) = -4C_4 \cos 2t - 4C_5 \sin 2t + \frac{1}{4}$$

Now apply the initial conditions to determine C_1, C_4, and C_5.

$$y(0) = C_1 + C_4 = 0$$
$$y'(0) = 2C_5 = 0$$
$$y''(0) = -4C_4 + \frac{1}{4} = 1$$

Solving this system of equations yields $C_1 = 3/16$, $C_4 = -3/16$, and $C_5 = 0$. Therefore,

$$y(t) = \frac{3}{16} - \frac{3}{16}\cos 2t + \frac{t^2}{8}.$$