# Problem 4

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' + y' = \sec t, \qquad -\pi/2 < t < \pi/2$$

---

**Solution**

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' + y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} + re^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^3 + r = 0$$

$$r(r^2 + 1) = 0$$

$$r = \{0, -i, i\}$$

Three solutions to equation (1) are then $y_c = e^0 = 1$ and $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution for $y_c$ is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1 + C_2 e^{-it} + C_3 e^{it} \\ &= C_1 + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \\ &= C_1 + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\ &= C_1 + C_4 \cos t + C_5 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' + y_p' = \sec t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t) + C_4(t)\cos t + C_5(t)\sin t$$

Substitute this formula into equation (2).

$$[C_1(t) + C_4(t)\cos t + C_5(t)\sin t]''' + [C_1(t) + C_4(t)\cos t + C_5(t)\sin t]' = \sec t$$

Evaluate the derivatives.

$$[C_1'(t) + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]'' \\ + [C_1'(t) + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t] = \sec t$$

If we set $C_1'(t) + C_4'(t)\cos t + C_5'(t)\sin t = 0$, then this equation simplifies to

$$[-C_4(t)\sin t + C_5(t)\cos t]'' + [-C_4(t)\sin t + C_5(t)\cos t] = \sec t$$

$$[-C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t]' + [-C_4(t)\sin t + C_5(t)\cos t] = \sec t.$$

If we set $-C_4'(t)\sin t + C_5'(t)\cos t = 0$, then this equation simplifies to

$$[-C_4(t)\cos t - C_5(t)\sin t]' + [-C_4(t)\sin t + C_5(t)\cos t] = \sec t$$

$$[-C_4'(t)\cos t + \cancel{C_4(t)\sin t} - C_5'(t)\sin t - \cancel{C_5(t)\cos t}] + [\cancel{-C_4(t)\sin t} + \cancel{C_5(t)\cos t}] = \sec t$$

$$-C_4'(t)\cos t - C_5'(t)\sin t = \sec t.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t) + C_4'(t)\cos t + C_5'(t)\sin t = 0 \tag{3}$$

$$-C_4'(t)\sin t + C_5'(t)\cos t = 0 \tag{4}$$

$$-C_4'(t)\cos t - C_5'(t)\sin t = \sec t \tag{5}$$

Start by solving equation (4) for $C_5'(t)$

$$C_5'(t) = \frac{\sin t}{\cos t} C_4'(t)$$

and then plugging it in to equation (5).

$$-C_4'(t)\cos t - \left[\frac{\sin t}{\cos t} C_4'(t)\right]\sin t = \sec t$$

Multiply both sides by $-\cos t$.

$$C_4'(t)\cos^2 t + C_4'(t)\sin^2 t = -1$$

$$C_4'(t)(\cos^2 t + \sin^2 t) = -1$$

$$C_4'(t) = -1$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_4(t) = -t$$

Substitute this back into equation (4) to get $C_5(t)$.

$$-C_4'(t)\sin t + C_5'(t)\cos t = 0 \quad \rightarrow \quad \sin t + C_5'(t)\cos t = 0 \quad \rightarrow \quad C_5'(t) = -\frac{\sin t}{\cos t}$$

$$C_5'(t) = -\tan t$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_5(t) = \ln|\cos t|$$

Substitute this result along with $C_4(t)$ into equation (3) to obtain $C_1(t)$.

$$C_1'(t) + C_4'(t)\cos t + C_5'(t)\sin t = 0 \quad \rightarrow \quad C_1'(t) + (-1)\cos t + (-\tan t)\sin t = 0$$

$$\begin{aligned} C_1'(t) &= \cos t + \frac{\sin^2 t}{\cos t} \\ &= \frac{\cos^2 t + \sin^2 t}{\cos t} \\ &= \frac{1}{\cos t} \\ &= \sec t \end{aligned}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_1(t) = \ln|\sec t + \tan t|$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t) + C_4(t)\cos t + C_5(t)\sin t \\ &= \ln|\sec t + \tan t| + (-t)\cos t + (\ln|\cos t|)\sin t \\ &= \ln|\sec t + \tan t| + \sin t \ln|\cos t| - t\cos t \\ &= \ln(\sec t + \tan t) + \sin t \ln\cos t - t\cos t. \end{aligned}$$

Because $-\pi/2 < t < \pi/2$, the absolute value signs have been dropped. Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 + C_4\cos t + C_5\sin t + \ln(\sec t + \tan t) + \sin t \ln\cos t - t\cos t. \end{aligned}$$