Problem 12

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

$$y''' - y' = \csc t; \qquad y(\pi/2) = 2, \quad y'(\pi/2) = 1, \quad y''(\pi/2) = -1$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - re^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2 + C_3e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = \csc t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{-t} + C_2(t) + C_3(t)e^t$$

Substitute this formula into equation (2).

$$[C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]''' - [C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]' = \csc t$$

Evaluate the derivatives.

$$[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t]'' - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t] = \csc t$$

If we set $C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0$, then this equation simplifies to

$$[-C_1(t)e^{-t} + C_3(t)e^t]'' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t$$

$$[-C_1'(t)e^{-t} + C_1(t)e^{-t} + C_3'(t)e^t + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t.$$

If we set $-C_1'(t)e^{-t} + C_3'(t)e^t = 0$, then this equation simplifies to

$$[C_1(t)e^{-t} + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t$$

$$[C_1'(t)e^{-t} - \cancel{C_1(t)e^{-t}} + C_3'(t)e^t + \cancel{C_3(t)e^t}] - [-\cancel{C_1(t)e^{-t}} + \cancel{C_3(t)e^t}] = \csc t$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = \csc t.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \tag{3}$$

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \tag{4}$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = \csc t \tag{5}$$

Start by solving equation (4) for $C_1'(t)$.

$$C_1'(t) = e^{2t}C_3'(t)$$

and then plugging it in to equation (5).

$$[e^{2t}C_3'(t)]e^{-t} + C_3'(t)e^t = \csc t$$

$$2C_3'(t)e^t = \csc t$$

$$C_3'(t) = \frac{1}{2}e^{-t}\csc t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_3(t) = \int^t \frac{1}{2}e^{-s}\csc s\,ds$$

Substitute this back into equation (4) to get $C_1(t)$.

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \quad \to \quad -C_1'(t)e^{-t} + \left[\frac{1}{2}e^{-t}\csc t\right]e^t = 0 \quad \to \quad -C_1'(t)e^{-t} + \frac{1}{2}\csc t = 0$$

$$C_1'(t) = \frac{1}{2}e^t\csc t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1(t) = \int^t \frac{1}{2}e^s\csc s\,ds$$

Substitute this result along with $C_3(t)$ into equation (3) to obtain $C_2(t)$.

$$C_1'(t)e^{-t}+C_2'(t)+C_3'(t)e^t = 0 \quad \to \quad \left[\frac{1}{2}e^t\csc t\right]e^{-t}+C_2'(t)+\left[\frac{1}{2}e^{-t}\csc t\right]e^t = 0 \quad \to \quad \frac{1}{2}\csc t+C_2'(t)+\frac{1}{2}\csc t = 0$$

$$C_2'(t) = -\csc t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2(t) = \ln|\csc t + \cot t|$$

The particular solution is then

$$\begin{aligned}
y_p(t) &= C_1(t)e^{-t} + C_2(t) + C_3(t)e^t \\
&= \left(\int^t \frac{1}{2}e^s \csc s\, ds\right) e^{-t} + \ln|\csc t + \cot t| + \left(\int^t \frac{1}{2}e^{-s} \csc s\, ds\right) e^t \\
&= \frac{1}{2}e^{-t}\int^t e^s \csc s\, ds + \ln|\csc t + \cot t| + \frac{1}{2}e^t \int^t e^{-s}\csc s\, ds,
\end{aligned}$$

and the general solution is

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_1 e^{-t} + C_2 + C_3 e^t + \frac{1}{2}e^{-t}\int^t e^s \csc s\, ds + \ln|\csc t + \cot t| + \frac{1}{2}e^t \int^t e^{-s}\csc s\, ds.
\end{aligned}$$

Differentiate it with respect to t twice.

$$y'(t) = -C_1e^{-t} + C_3e^t - \frac{1}{2}e^{-t}\int^t e^s \csc s\, ds + \cancel{\frac{1}{2}e^{-t}(e^t \csc t)} - \cancel{\csc t} + \frac{1}{2}e^t\int^t e^{-s}\csc s\, ds + \cancel{\frac{1}{2}e^t(e^{-t}\csc t)}$$

$$y''(t) = C_1e^{-t} + C_3e^t + \frac{1}{2}e^{-t}\int^t e^s \csc s\, ds - \cancel{\frac{1}{2}e^{-t}(e^t \csc t)} + \frac{1}{2}e^t\int^t e^{-s}\csc s\, ds + \cancel{\frac{1}{2}e^t(e^{-t}\csc t)}$$

Now apply the initial conditions to determine C_1, C_2, and C_3. Since they are given at $t = \pi/2$, let $\pi/2$ be the lower limit of integration in all integrals.

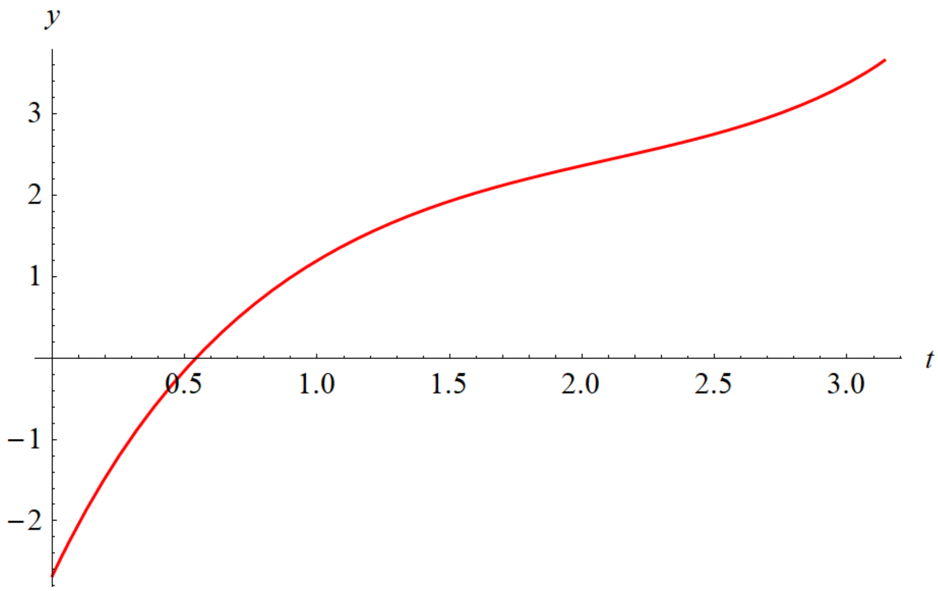
$$\begin{aligned}
y(\pi/2) &= C_1e^{-\pi/2} + C_2 + C_3e^{\pi/2} = 2 \\
y'(\pi/2) &= -C_1e^{-\pi/2} + C_3e^{\pi/2} = 1 \\
y''(\pi/2) &= C_1e^{-\pi/2} + C_3e^{\pi/2} = -1
\end{aligned}$$

Solving this system of equations yields $C_1 = -e^{\pi/2}$, $C_2 = 3$, and $C_3 = 0$.

$$\begin{aligned}
y(t) &= -e^{\pi/2}e^{-t} + 3 + \frac{1}{2}e^{-t}\int_{\pi/2}^t e^s \csc s\, ds + \ln|\csc t + \cot t| + \frac{1}{2}e^t\int_{\pi/2}^t e^{-s}\csc s\, ds \\
&= 3 - e^{\pi/2 - t} + \ln|\csc t + \cot t| + \frac{1}{2}\int_{\pi/2}^t (e^{-t+s}\csc s + e^{t-s}\csc s)\, ds \\
&= 3 - e^{\pi/2 - t} + \ln|\csc t + \cot t| + \int_{\pi/2}^t \frac{e^{-(t-s)} + e^{t-s}}{2}\csc s\, ds
\end{aligned}$$

Therefore,

$$y(t) = 3 - e^{\pi/2 - t} + \ln|\csc t + \cot t| + \int_{\pi/2}^t \cosh(t-s)\csc s\, ds.$$


y
3
2
1
−1
−2
0.5
1.0
1.5
2.0
2.5
3.0
t