Problem 15

Find a formula involving integrals for a particular solution of the differential equation

$$y^{(4)} - y = g(t).$$

Hint: The functions $\sin t$, $\cos t$, $\sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt} \quad \rightarrow \quad y_c''' = r^3 e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^4 - 1 = 0$$

$$(r^2 + 1)(r^2 - 1) = 0$$

$$r = \{-i, i, -1, 1\}$$

Four solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$ and $y_c = e^{-t}$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 e^{-it} + C_2 e^{it} + C_3 e^{-t} + C_4 e^t \\ &= C_1(\cos t - i \sin t) + C_2(\cos t + i \sin t) + C_3(\cosh t - \sinh t) + C_4(\cosh t + \sinh t) \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + (C_3 + C_4)\cosh t + (-C_3 + C_4)\sinh t \\ &= C_5 \cos t + C_6 \sin t + C_7 \cosh t + C_8 \sinh t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} - y_p = g(t). \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t$$

Substitute this formula into equation (2).

$$[C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t]^{(4)} \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

Evaluate the derivatives.

$$[C_5'(t)\cos t - C_5(t)\sin t + C_6'(t)\sin t + C_6(t)\cos t + C_7'(t)\cosh t + C_7(t)\sinh t + C_8'(t)\sinh t + C_8(t)\cosh t]''' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

If we set $C_5'(t)\cos t + C_6'(t)\sin t + C_7'(t)\cosh t + C_8'(t)\sinh t = 0$, then this equation simplifies to

$$[-C_5(t)\sin t + C_6(t)\cos t + C_7(t)\sinh t + C_8(t)\cosh t]''' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

$$[-C_5'(t)\sin t - C_5(t)\cos t + C_6'(t)\cos t - C_6(t)\sin t + C_7'(t)\sinh t + C_7(t)\cosh t + C_8'(t)\cosh t + C_8(t)\sinh t]'' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t).$$

If we set $-C_5'(t)\sin t + C_6'(t)\cos t + C_7'(t)\sinh t + C_8'(t)\cosh t = 0$, then this equation simplifies to

$$[-C_5(t)\cos t - C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t]'' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

$$[-C_5'(t)\cos t + C_5(t)\sin t - C_6'(t)\sin t - C_6(t)\cos t + C_7'(t)\cosh t + C_7(t)\sinh t + C_8'(t)\sinh t + C_8(t)\cosh t]' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t).$$

If we set $-C_5'(t)\cos t - C_6'(t)\sin t + C_7'(t)\cosh t + C_8'(t)\sinh t = 0$, then this equation simplifies to

$$[C_5(t)\sin t - C_6(t)\cos t + C_7(t)\sinh t + C_8(t)\cosh t]' \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

$$[C_5'(t)\sin t + C_5(t)\cos t - C_6'(t)\cos t + C_6(t)\sin t + C_7'(t)\sinh t + C_7(t)\cosh t + C_8'(t)\cosh t + C_8(t)\sinh t] \\ - [C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t] = g(t)$$

$$C_5'(t)\sin t - C_6'(t)\cos t + C_7'(t)\sinh t + C_8'(t)\cosh t = g(t).$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_5'(t)\cos t + C_6'(t)\sin t + C_7'(t)\cosh t + C_8'(t)\sinh t = 0 \tag{3}$$
$$-C_5'(t)\sin t + C_6'(t)\cos t + C_7'(t)\sinh t + C_8'(t)\cosh t = 0 \tag{4}$$
$$-C_5'(t)\cos t - C_6'(t)\sin t + C_7'(t)\cosh t + C_8'(t)\sinh t = 0 \tag{5}$$
$$C_5'(t)\sin t - C_6'(t)\cos t + C_7'(t)\sinh t + C_8'(t)\cosh t = g(t) \tag{6}$$

Add the respective sides of equations (3) and (5). Also, add the respective sides of equations (4) and (6). Doing so eliminates $C_5'(t)$ and $C_6'(t)$.

$$\begin{aligned} 2C_7'(t)\cosh t + 2C_8'(t)\sinh t &= 0 \\ 2C_7'(t)\sinh t + 2C_8'(t)\cosh t &= g(t) \end{aligned}$$

Solve this first equation for $C_7'(t)$

$$C_7'(t) = -\frac{\sinh t}{\cosh t}C_8'(t) \tag{7}$$

and then plug it into the second equation.

$$2\left[-\frac{\sinh t}{\cosh t}C_8'(t)\right]\sinh t + 2C_8'(t)\cosh t = g(t)$$

Divide both sides by 2 and factor $C_8'(t)$.

$$C_8'(t)\left(-\frac{\sinh^2 t}{\cosh t} + \cosh t\right) = \frac{1}{2}g(t)$$

$$C_8'(t)\left(\frac{-\sinh^2 t + \cosh^2 t}{\cosh t}\right) = \frac{1}{2}g(t)$$

$$C_8'(t)\left(\frac{1}{\cosh t}\right) = \frac{1}{2}g(t)$$

$$C_8'(t) = \frac{1}{2}g(t)\cosh t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_8(t) = \int^t \frac{1}{2}g(s)\cosh s\,ds$$

Substitute this result into equation (7) to get $C_7(t)$.

$$\begin{aligned} C_7'(t) &= -\frac{\sinh t}{\cosh t}\left[\frac{1}{2}g(t)\cosh t\right] \\ &= -\frac{1}{2}g(t)\sinh t \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_7(t) = \int^t -\frac{1}{2}g(s)\sinh s\,ds$$

Subtract the respective sides of equation (5) from those of equation (3). Also, subtract the respective sides of equation (6) from those of equation (4). Doing so eliminates $C_7'(t)$ and $C_8'(t)$.

$$\begin{aligned} 2C_5'(t)\cos t + 2C_6'(t)\sin t &= 0 \\ -2C_5'(t)\sin t + 2C_6'(t)\cos t &= -g(t) \end{aligned}$$

Solve this first equation for $C_5'(t)$

$$C_5'(t) = -\frac{\sin t}{\cos t}C_6'(t) \tag{8}$$

and then plug it into the second equation.

$$-2\left[-\frac{\sin t}{\cos t}C_6'(t)\right]\sin t + 2C_6'(t)\cos t = -g(t)$$

Divide both sides by 2 and factor $C_6'(t)$.

$$C_6'(t)\left(\frac{\sin^2 t}{\cos t} + \cos t\right) = -\frac{1}{2}g(t)$$
$$C_6'(t)\left(\frac{\sin^2 t + \cos^2 t}{\cos t}\right) = -\frac{1}{2}g(t)$$
$$C_6'(t)\left(\frac{1}{\cos t}\right) = -\frac{1}{2}g(t)$$
$$C_6'(t) = -\frac{1}{2}g(t)\cos t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_6(t) = \int^t -\frac{1}{2}g(s)\cos s\,ds$$

Substitute this result for $C_6'(t)$ into equation (8) to get $C_5(t)$.

$$\begin{aligned} C_5'(t) &= -\frac{\sin t}{\cos t}\left[-\frac{1}{2}g(t)\cos t\right] \\ &= \frac{1}{2}g(t)\sin t \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_5(t) = \int^t \frac{1}{2}g(s)\sin s\,ds$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_5(t)\cos t + C_6(t)\sin t + C_7(t)\cosh t + C_8(t)\sinh t \\ &= \left[\int^t \frac{1}{2}g(s)\sin s\,ds\right]\cos t + \left[\int^t -\frac{1}{2}g(s)\cos s\,ds\right]\sin t \\ &\qquad + \left[\int^t -\frac{1}{2}g(s)\sinh s\,ds\right]\cosh t + \left[\int^t \frac{1}{2}g(s)\cosh s\,ds\right]\sinh t \\ &= \frac{1}{2}\int^t (\cos t\sin s - \sin t\cos s - \cosh t\sinh s + \sinh t\cosh s)g(s)\,ds \\ &= \frac{1}{2}\int^t [-(\sin t\cos s - \cos t\sin s) + (\sinh t\cosh s - \cosh t\sinh s)]g(s)\,ds \\ &= \frac{1}{2}\int^t [-\sin(t-s) + \sinh(t-s)]g(s)\,ds, \end{aligned}$$

and the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_5\cos t + C_6\sin t + C_7\cosh t + C_8\sinh t + \frac{1}{2}\int^t [-\sin(t-s) + \sinh(t-s)]g(s)\,ds. \end{aligned}$$