

Problem 10

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y^{(4)} + 2y'' + y = \sin t; \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 1$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c'' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \rightarrow y_c' = r e^{rt} \rightarrow y_c'' = r^2 e^{rt} \rightarrow y_c''' = r^3 e^{rt} \rightarrow y_c^{(4)} = r^4 e^{rt}$$

Substitute these expressions into the ODE.

$$r^4 e^{rt} + 2(r^2 e^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2 + 1)^2 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of t : $y_c = te^{-it}$ and $y_c = te^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1 e^{-it} + C_2 e^{it} + C_3 t e^{-it} + C_4 t e^{it} \\ &= C_1 (\cos t - i \sin t) + C_2 (\cos t + i \sin t) + C_3 t (\cos t - i \sin t) + C_4 t (\cos t + i \sin t) \\ &= (C_1 + C_2) \cos t + (-iC_1 + iC_2) \sin t + t(C_3 + C_4) \cos t + t(-iC_3 + iC_4) \sin t \\ &= C_5 \cos t + C_6 \sin t + C_7 t \cos t + C_8 t \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p'' + y_p = \sin t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_5(t) \cos t + C_6(t) \sin t + C_7(t) t \cos t + C_8(t) t \sin t$$

Substitute this formula into equation (2).

$$\begin{aligned}
 & [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t]^{(4)} \\
 & \quad + 2[C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t]'' \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t
 \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}
 & [C_5'(t) \cos t - C_5(t) \sin t + C_6'(t) \sin t + C_6(t) \cos t + C_7'(t)t \cos t + C_7(t) \cos t - C_7(t)t \sin t \\
 & \quad + C_8'(t)t \sin t + C_8(t) \sin t + C_8(t)t \cos t]''' + 2[C_5'(t) \cos t - C_5(t) \sin t + C_6'(t) \sin t + C_6(t) \cos t \\
 & \quad + C_7'(t)t \cos t + C_7(t) \cos t - C_7(t)t \sin t + C_8'(t)t \sin t + C_8(t) \sin t + C_8(t)t \cos t]' \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t
 \end{aligned}$$

If we set $C_5'(t) \cos t + C_6'(t) \sin t + C_7'(t)t \cos t + C_8'(t)t \sin t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [-C_5(t) \sin t + C_6(t) \cos t + C_7(t) \cos t - C_7(t)t \sin t + C_8(t) \sin t + C_8(t)t \cos t]''' \\
 & \quad + 2[-C_5(t) \sin t + C_6(t) \cos t + C_7(t) \cos t - C_7(t)t \sin t + C_8(t) \sin t + C_8(t)t \cos t]' \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t
 \end{aligned}$$

$$\begin{aligned}
 & [-C_5'(t) \sin t - C_5(t) \cos t + C_6'(t) \cos t - C_6(t) \sin t + C_7'(t) \cos t - C_7(t) \sin t - C_7'(t)t \sin t - C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + C_8'(t) \sin t + C_8(t) \cos t + C_8'(t)t \cos t + C_8(t) \cos t - C_8(t)t \sin t]'' \\
 & + 2[-C_5'(t) \sin t - C_5(t) \cos t + C_6'(t) \cos t - C_6(t) \sin t + C_7'(t) \cos t - C_7(t) \sin t - C_7'(t)t \sin t - C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + C_8'(t) \sin t + C_8(t) \cos t + C_8'(t)t \cos t + C_8(t) \cos t - C_8(t)t \sin t] \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t.
 \end{aligned}$$

If we set $-C_5'(t) \sin t + C_6'(t) \cos t + C_7'(t) \cos t - C_7'(t)t \sin t + C_8'(t) \sin t + C_8'(t)t \cos t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [-C_5(t) \cos t - C_6(t) \sin t - 2C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + 2C_8(t) \cos t - C_8(t)t \sin t]'' \\
 & \quad + 2[-C_5(t) \cos t - C_6(t) \sin t - 2C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + 2C_8(t) \cos t - C_8(t)t \sin t] \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t
 \end{aligned}$$

$$\begin{aligned}
 & [-C_5'(t) \cos t + C_5(t) \sin t - C_6'(t) \sin t - C_6(t) \cos t - 2C_7'(t) \sin t - 2C_7(t) \cos t - C_7'(t)t \cos t \\
 & \quad - C_7(t) \cos t + C_7(t)t \sin t + 2C_8'(t) \cos t - 2C_8(t) \sin t - C_8'(t)t \sin t - C_8(t) \sin t - C_8(t)t \cos t]' \\
 & \quad + 2[-C_5(t) \cos t - C_6(t) \sin t - 2C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + 2C_8(t) \cos t - C_8(t)t \sin t] \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t.
 \end{aligned}$$

If we set $-C_5'(t) \cos t - C_6'(t) \sin t - 2C_7'(t) \sin t - C_7'(t)t \cos t + 2C_8'(t) \cos t - C_8'(t)t \sin t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [C_5(t) \sin t - C_6(t) \cos t - 2C_7(t) \cos t - C_7(t) \cos t + C_7(t)t \sin t - 2C_8(t) \sin t - C_8(t) \sin t - C_8(t)t \cos t]' \\
 & \quad + 2[-C_5(t) \cos t - C_6(t) \sin t - 2C_7(t) \sin t \\
 & \quad - C_7(t)t \cos t + 2C_8(t) \cos t - C_8(t)t \sin t] \\
 & \quad + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t
 \end{aligned}$$

$$\begin{aligned}
 & [C'_5(t) \sin t + C_5(t) \cos t - C'_6(t) \cos t + C_6(t) \sin t - 2C'_7(t) \cos t + 2C_7(t) \sin t - C'_7(t) \cos t + C_7(t) \sin t \\
 & + C'_7(t)t \sin t + C_7(t) \sin t + C_7(t)t \cos t - 2C'_8(t) \sin t - 2C_8(t) \cos t - C'_8(t) \sin t \\
 & - C_8(t) \cos t - C'_8(t)t \cos t - C_8(t) \cos t + C_8(t)t \sin t] \\
 & + 2[-C_5(t) \cos t - C_6(t) \sin t - 2C_7(t) \sin t \\
 & - C_7(t)t \cos t + 2C_8(t) \cos t - C_8(t)t \sin t] \\
 & + [C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t] = \sin t \\
 & C'_5(t) \sin t - C'_6(t) \cos t + C'_7(t)(t \sin t - 3 \cos t) + C'_8(t)(-3 \sin t - t \cos t) = \sin t
 \end{aligned}$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C'_5(t) \cos t + C'_6(t) \sin t + C'_7(t)t \cos t + C'_8(t)t \sin t = 0 \quad (3)$$

$$-C'_5(t) \sin t + C'_6(t) \cos t + C'_7(t)(\cos t - t \sin t) + C'_8(t)(\sin t + t \cos t) = 0 \quad (4)$$

$$-C'_5(t) \cos t - C'_6(t) \sin t + C'_7(t)(-2 \sin t - t \cos t) + C'_8(t)(2 \cos t - t \sin t) = 0 \quad (5)$$

$$C'_5(t) \sin t - C'_6(t) \cos t + C'_7(t)(t \sin t - 3 \cos t) + C'_8(t)(-3 \sin t - t \cos t) = \sin t \quad (6)$$

Add the respective sides of equations (3) and (5) together, and add the respective sides of equations (4) and (6) together. Doing so eliminates $C'_5(t)$ and $C'_6(t)$.

$$C'_7(t)(-2 \sin t) + C'_8(t)(2 \cos t) = 0$$

$$C'_7(t)(-2 \cos t) + C'_8(t)(-2 \sin t) = \sin t$$

Solve this first equation for $C'_8(t)$

$$C'_8(t) = \frac{\sin t}{\cos t} C'_7(t) \quad (7)$$

and plug it into the second equation.

$$C'_7(t)(-2 \cos t) + \left[\frac{\sin t}{\cos t} C'_7(t) \right] (-2 \sin t) = \sin t$$

$$C'_7(t) \left(\cos t + \frac{\sin t}{\cos t} \sin t \right) = -\frac{1}{2} \sin t$$

$$C'_7(t) \left(\frac{\cos^2 t + \sin^2 t}{\cos t} \right) = -\frac{1}{2} \sin t$$

$$C'_7(t) \left(\frac{1}{\cos t} \right) = -\frac{1}{2} \sin t$$

$$\begin{aligned}
 C'_7(t) &= -\frac{1}{2} \sin t \cos t \\
 &= -\frac{1}{4} \sin 2t
 \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_7(t) = \frac{1}{8} \cos 2t$$

Substitute this back into equation (7) to get $C_8(t)$.

$$\begin{aligned} C_8'(t) &= \frac{\sin t}{\cos t} \left(-\frac{1}{2} \sin t \cos t \right) \\ &= -\frac{1}{2} \sin^2 t \\ &= -\frac{1}{4} (1 - \cos 2t) \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_8(t) = \frac{1}{8} (\sin 2t - 2t)$$

Substitute the results for $C_7'(t)$ and $C_8'(t)$ into equations (3) and (4).

$$\begin{aligned} C_5'(t) \cos t + C_6'(t) \sin t + \left(-\frac{1}{2} \sin t \cos t \right) t \cos t + \left(-\frac{1}{2} \sin^2 t \right) t \sin t &= 0 \\ -C_5'(t) \sin t + C_6'(t) \cos t + \left(-\frac{1}{2} \sin t \cos t \right) (\cos t - t \sin t) + \left(-\frac{1}{2} \sin^2 t \right) (\sin t + t \cos t) &= 0 \end{aligned}$$

Simplify these equations.

$$\begin{aligned} C_5'(t) \cos t + C_6'(t) \sin t &= \frac{1}{2} t \sin t \\ -C_5'(t) \sin t + C_6'(t) \cos t &= \frac{1}{2} \sin t \end{aligned} \tag{8}$$

Solve this second equation for $C_6'(t)$

$$C_6'(t) = \frac{1}{\cos t} \left[\frac{1}{2} \sin t + C_5'(t) \sin t \right]$$

and plug it into the first equation.

$$C_5'(t) \cos t + \frac{1}{\cos t} \left[\frac{1}{2} \sin t + C_5'(t) \sin t \right] \sin t = \frac{1}{2} t \sin t$$

$$C_5'(t) \left(\cos t + \frac{\sin^2 t}{\cos t} \right) + \frac{1}{2} \frac{\sin^2 t}{\cos t} = \frac{1}{2} t \sin t$$

$$C_5'(t) \left(\frac{\cos^2 t + \sin^2 t}{\cos t} \right) = \frac{1}{2} t \sin t - \frac{1}{2} \frac{\sin^2 t}{\cos t}$$

$$C_5'(t) \left(\frac{1}{\cos t} \right) = \frac{1}{2} t \sin t - \frac{1}{2} \frac{\sin^2 t}{\cos t}$$

$$\begin{aligned} C_5'(t) &= \frac{1}{2} t \sin t \cos t - \frac{1}{2} \sin^2 t \\ &= \frac{1}{4} t \sin 2t - \frac{1}{4} (1 - \cos 2t) \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$\begin{aligned} C_5(t) &= \frac{1}{16}(\sin 2t - 2t \cos 2t) - \frac{t}{4} + \frac{1}{8} \sin 2t \\ &= \frac{1}{16}(3 \sin 2t - 2t \cos 2t) - \frac{t}{4} \end{aligned}$$

Substitute this result into equation (8) to determine $C_6(t)$.

$$\begin{aligned} \left(\frac{1}{2}t \sin t \cos t - \frac{1}{2} \sin^2 t\right) \cos t + C_6'(t) \sin t &= \frac{1}{2}t \sin t \\ \left(\frac{1}{2}t \cos t - \frac{1}{2} \sin t\right) \cos t + C_6'(t) &= \frac{1}{2}t \\ \frac{1}{2}t \cos^2 t - \frac{1}{2} \sin t \cos t + C_6'(t) &= \frac{1}{2}t \\ C_6'(t) &= \frac{1}{2}t(1 - \cos^2 t) + \frac{1}{2} \sin t \cos t \\ &= \frac{1}{2}t \sin^2 t + \frac{1}{4} \sin 2t \\ &= \frac{1}{4}t(1 - \cos 2t) + \frac{1}{4} \sin 2t \\ &= \frac{t}{4} - \frac{t}{4} \cos 2t + \frac{1}{4} \sin 2t \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$\begin{aligned} C_6(t) &= \frac{t^2}{8} - \frac{1}{16}(2t \sin 2t + \cos 2t) - \frac{1}{8} \cos 2t \\ &= \frac{t^2}{8} - \frac{1}{16}(2t \sin 2t + 3 \cos 2t) \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_5(t) \cos t + C_6(t) \sin t + C_7(t)t \cos t + C_8(t)t \sin t \\ &= \left[\frac{1}{16}(3 \sin 2t - 2t \cos 2t) - \frac{t}{4}\right] \cos t + \left[\frac{t^2}{8} - \frac{1}{16}(2t \sin 2t + 3 \cos 2t)\right] \sin t \\ &\quad + \left(\frac{1}{8} \cos 2t\right) t \cos t + \left[\frac{1}{8}(\sin 2t - 2t)\right] t \sin t \\ &= -\frac{t}{4} \cos t + \frac{3}{16} \sin t - \frac{t^2}{8} \sin t, \end{aligned}$$

and the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_5 \cos t + C_6 \sin t + C_7 t \cos t + C_8 t \sin t - \frac{t}{4} \cos t + \frac{3}{16} \sin t - \frac{t^2}{8} \sin t \\ &= C_5 \cos t + \left(C_6 + \frac{3}{16}\right) \sin t + \left(C_7 - \frac{1}{4}\right) t \cos t + C_8 t \sin t - \frac{t^2}{8} \sin t \\ &= C_5 \cos t + C_9 \sin t + C_{10} t \cos t + C_8 t \sin t - \frac{t^2}{8} \sin t. \end{aligned}$$

Differentiate it with respect to t three times.

$$y'(t) = -C_5 \sin t + C_9 \cos t + C_{10} \cos t - C_{10}t \sin t + C_8 \sin t + C_8 t \cos t - \frac{t}{4} \sin t - \frac{t^2}{8} \cos t$$

$$y''(t) = -C_5 \cos t - C_9 \sin t - C_{10} \sin t - C_{10} \sin t - C_{10}t \cos t + C_8 \cos t + C_8 \cos t \\ - C_8 t \sin t - \frac{1}{4} \sin t - \frac{t}{4} \cos t - \frac{t}{4} \cos t + \frac{t^2}{8} \sin t$$

$$y'''(t) = C_5 \sin t - C_9 \cos t - C_{10} \cos t - C_{10} \cos t - C_{10} \cos t + C_{10}t \sin t - 2C_8 \sin t \\ - C_8 \sin t - C_8 t \cos t - \frac{1}{4} \cos t - \frac{1}{4} \cos t + \frac{t}{4} \sin t - \frac{1}{4} \cos t + \frac{t}{4} \sin t + \frac{t}{4} \sin t + \frac{t^2}{8} \cos t$$

Now apply the initial conditions to determine C_1 , C_4 , and C_5 .

$$y(0) = C_5 = 2$$

$$y'(0) = C_9 + C_{10} = 0$$

$$y''(0) = -C_5 + C_8 + C_8 = -1$$

$$y'''(0) = -C_9 - C_{10} - C_{10} - C_{10} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 1$$

Solving this system of equations yields $C_5 = 2$, $C_8 = 1/2$, $C_9 = 7/8$, and $C_{10} = -7/8$. Therefore,

$$y(t) = 2 \cos t + \frac{7}{8} \sin t - \frac{7}{8}t \cos t + \frac{1}{2}t \sin t - \frac{t^2}{8} \sin t.$$



