

Problem 11

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

$$y''' - y'' + y' - y = \sec t; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = 1$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c'' + y_c' - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \rightarrow y_c' = r e^{rt} \rightarrow y_c'' = r^2 e^{rt} \rightarrow y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - r^2 e^{rt} + r e^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^3 - r^2 + r - 1 = 0$$

$$(r - 1)(r^2 + 1) = 0$$

$$r = \{1, -i, i\}$$

Three solutions to equation (1) are then $y_c = e^t$ and $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1 e^t + C_2 e^{-it} + C_3 e^{it} \\ &= C_1 e^t + C_2 (\cos t - i \sin t) + C_3 (\cos t + i \sin t) \\ &= C_1 e^t + (C_2 + C_3) \cos t + (-iC_2 + iC_3) \sin t \\ &= C_1 e^t + C_4 \cos t + C_5 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p'' + y_p' - y_p = \sec t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t$$

Substitute this formula into equation (2).

$$\begin{aligned} &[C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]''' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]'' \\ &+ [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t] = \sec t \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}
 & [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]'' \\
 & \quad - [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]' \\
 & \quad + [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t] \\
 & \quad \quad - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t
 \end{aligned}$$

If we set $C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]'' - [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]' \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t
 \end{aligned}$$

$$\begin{aligned}
 & [C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t]' \\
 & \quad - [C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t] \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t.
 \end{aligned}$$

If we set $C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t]' - [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t] \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t
 \end{aligned}$$

$$\begin{aligned}
 & [C_1'(t)e^t + \cancel{C_1(t)e^t} - C_4'(t)\cos t + \cancel{C_4(t)\sin t} - C_5'(t)\sin t - C_5(t)\cos t] \\
 & \quad - [\cancel{C_1(t)e^t} - \cancel{C_4(t)\cos t} - C_5(t)\sin t] \\
 & \quad + [\cancel{C_1(t)e^t} - \cancel{C_4(t)\sin t} + C_5(t)\cos t] - [\cancel{C_1(t)e^t} + \cancel{C_4(t)\cos t} + C_5(t)\sin t] = \sec t \\
 & \quad \quad C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = \sec t.
 \end{aligned}$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0 \quad (3)$$

$$C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0 \quad (4)$$

$$C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = \sec t \quad (5)$$

Add the respective sides of equations (3) and (5) together.

$$2C_1'(t)e^t = \sec t$$

$$C_1'(t) = \frac{1}{2}e^{-t}\sec t$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_1(t) = \int \frac{1}{2}e^{-s}\sec s \, ds$$

Subtract the respective sides of equation (4) from those of equation (3). Also, subtract the respective sides of equation (5) from those of equation (3).

$$\begin{aligned} C_4'(t)(\cos t + \sin t) + C_5'(\sin t - \cos t) &= 0 \\ 2C_4'(t) \cos t + 2C_5'(t) \sin t &= -\sec t \end{aligned}$$

Solve this first equation for $C_4'(t)$

$$C_4'(t) = \frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \quad (6)$$

and then plug it into the second equation.

$$2 \left[\frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \right] \cos t + 2C_5'(t) \sin t = -\sec t$$

Divide both sides by 2 and factor $C_5'(t)$.

$$\begin{aligned} C_5'(t) \left(\frac{\cos t - \sin t}{\cos t + \sin t} \cos t + \sin t \right) &= -\frac{1}{2} \sec t \\ C_5'(t) \left[\frac{\cos t - \sin t}{\cos t + \sin t} \cos t + \frac{\sin t(\cos t + \sin t)}{\cos t + \sin t} \right] &= -\frac{1}{2} \sec t \\ C_5'(t) \left(\frac{\cos^2 t - \sin t \cos t + \sin t \cos t + \sin^2 t}{\cos t + \sin t} \right) &= -\frac{1}{2} \sec t \\ C_5'(t) \left(\frac{1}{\cos t + \sin t} \right) &= -\frac{1}{2} \sec t \end{aligned}$$

$$\begin{aligned} C_5'(t) &= -\frac{1}{2} \sec t (\cos t + \sin t) \\ &= -\frac{1}{2} - \frac{1}{2} \tan t \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$\begin{aligned} C_5(t) &= -\frac{t}{2} - \frac{1}{2} \ln |\sec t| \\ &= -\frac{t}{2} + \frac{1}{2} \ln |\cos t| \end{aligned}$$

Now substitute the previous equation into equation (6).

$$\begin{aligned} C_4'(t) &= \frac{\cos t - \sin t}{\cos t + \sin t} \left[-\frac{1}{2} \sec t (\cos t + \sin t) \right] \\ &= -\frac{1}{2} \sec t (\cos t - \sin t) \\ &= -\frac{1}{2} + \frac{1}{2} \tan t \end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$\begin{aligned} C_4(t) &= -\frac{t}{2} + \frac{1}{2} \ln |\sec t| \\ &= -\frac{t}{2} - \frac{1}{2} \ln |\cos t| \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t \\ &= \left[\int^t \frac{1}{2} e^{-s} \sec s \, ds \right] e^t + \left[-\frac{t}{2} - \frac{1}{2} \ln |\cos t| \right] \cos t + \left[-\frac{t}{2} + \frac{1}{2} \ln |\cos t| \right] \sin t \\ &= \frac{1}{2} e^t \int^t e^{-s} \sec s \, ds - \frac{t}{2} \cos t - \frac{1}{2} \cos t \ln |\cos t| - \frac{t}{2} \sin t + \frac{1}{2} \sin t \ln |\cos t|, \end{aligned}$$

and the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 e^t + C_4 \cos t + C_5 \sin t + \frac{1}{2} e^t \int^t e^{-s} \sec s \, ds - \frac{t}{2} \cos t - \frac{1}{2} \cos t \ln |\cos t| - \frac{t}{2} \sin t + \frac{1}{2} \sin t \ln |\cos t|. \end{aligned}$$

Differentiate this solution with respect to t twice.

$$\begin{aligned} y'(t) &= C_1 e^t - C_4 \sin t + C_5 \cos t + \frac{1}{2} e^t \int^t e^{-s} \sec s \, ds + \frac{1}{2} e^t (e^{-t} \sec t) - \frac{1}{2} \cos t + \frac{t}{2} \sin t + \frac{1}{2} \sin t \ln |\cos t| \\ &\quad + \frac{1}{2} \cos t \tan t - \frac{1}{2} \sin t - \frac{t}{2} \cos t + \frac{1}{2} \cos t \ln |\cos t| + \frac{1}{2} \sin t (-\tan t) \\ &= C_1 e^t - C_4 \sin t + C_5 \cos t + \frac{1}{2} e^t \int^t e^{-s} \sec s \, ds + \frac{1}{2} \sec t - \frac{1}{2} \cos t + \frac{t}{2} \sin t + \frac{1}{2} \sin t \ln |\cos t| \\ &\quad - \frac{t}{2} \cos t + \frac{1}{2} \cos t \ln |\cos t| - \frac{1}{2} \sin t \tan t \\ y''(t) &= C_1 e^t - C_4 \cos t - C_5 \sin t + \frac{1}{2} e^t \int^t e^{-s} \sec s \, ds + \frac{1}{2} e^t (e^{-t} \sec t) + \frac{1}{2} \sec t \tan t + \frac{1}{2} \sin t + \frac{1}{2} \sin t \\ &\quad + \frac{t}{2} \cos t + \frac{1}{2} \cos t \ln |\cos t| + \frac{1}{2} \sin t (-\tan t) - \frac{1}{2} \cos t + \frac{t}{2} \sin t \\ &\quad - \frac{1}{2} \sin t \ln |\cos t| + \frac{1}{2} \cos t (-\tan t) - \frac{1}{2} \cos t \tan t - \frac{1}{2} \sin t (\sec^2 t) \end{aligned}$$

Now apply the initial conditions to determine C_1 , C_4 , and C_5 . Since they are given at $t = 0$, let 0 be the lower limit of integration in all integrals.

$$\begin{aligned} y(0) &= C_1 + C_4 = 2 \\ y'(0) &= C_1 + C_5 + \frac{1}{2} - \frac{1}{2} = -1 \\ y''(0) &= C_1 - C_4 + \frac{1}{2} - \frac{1}{2} = 1 \end{aligned}$$

Solving this system of equations yields $C_1 = 3/2$, $C_4 = 1/2$, and $C_5 = -5/2$. Therefore,

$$y(t) = \frac{3}{2} e^t + \frac{1}{2} \cos t - \frac{5}{2} \sin t + \frac{1}{2} e^t \int_0^t e^{-s} \sec s \, ds - \frac{t}{2} \cos t - \frac{1}{2} \cos t \ln |\cos t| - \frac{t}{2} \sin t + \frac{1}{2} \sin t \ln |\cos t|.$$

