Problem 13

Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \qquad x > 0,$$

determine a particular solution.

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(x)$ and $y_p(x)$, the complementary solution and the particular solution, respectively.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$x^3y_c''' + x^2y_c'' - 2xy_c' + 2y_c = 0 \tag{1}$$

Three solutions are $y_c = x$ and $y_c = x^2$ and $y_c = 1/x = x^{-1}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(x) = C_1x + C_2x^2 + C_3x^{-1}$$

On the other hand, the particular solution satisfies

$$x^3y_p''' + x^2y_p'' - 2xy_p' + 2y_p = 2x^4. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(x)$ to vary.

$$y_p(x) = C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}$$

Substitute this formula into equation (2).

$$\begin{aligned} x^3[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}]''' &+ x^2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}]'' \\ &- 2x[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}]' + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4 \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned} x^3[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^{-1} - C_3(x)x^{-2}]'' \\ + x^2[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^{-1} - C_3(x)x^{-2}]' \\ - 2x[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^{-1} - C_3(x)x^{-2}] \\ + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4 \end{aligned}$$

If we set $C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^{-1} = 0$, then this equation simplifies to

$$\begin{aligned} x^3[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}]'' &+ x^2[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}]' \\ &- 2x[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}] + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4 \end{aligned}$$

$$\begin{aligned} x^3[C_1'(x) + 2C_2'(x)x + 2C_2(x) - C_3'(x)x^{-2} + 2C_3(x)x^{-3}]' \\ + x^2[C_1'(x) + 2C_2'(x)x + 2C_2(x) - C_3'(x)x^{-2} + 2C_3(x)x^{-3}] \\ - 2x[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}] \\ + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4. \end{aligned}$$

If we set $C_1'(x) + 2C_2'(x)x - C_3'(x)x^{-2} = 0$, then this equation simplifies to

$$\begin{aligned} x^3[2C_2(x) + 2C_3(x)x^{-3}]' + x^2[2C_2(x) + 2C_3(x)x^{-3}] \\ - 2x[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}] + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4 \end{aligned}$$

$$\begin{aligned} x^3[2C_2'(x) + 2C_3'(x)x^{-3} - 6C_3(x)x^{-4}] + x^2[2C_2(x) + 2C_3(x)x^{-3}] \\ - 2x[C_1(x) + 2C_2(x)x - C_3(x)x^{-2}] + 2[C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1}] = 2x^4 \end{aligned}$$

$$2C_2'(x)x^3 + 2C_3'(x) = 2x^4.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^{-1} = 0 \tag{3}$$

$$C_1'(x) + 2C_2'(x)x - C_3'(x)x^{-2} = 0 \tag{4}$$

$$2C_2'(x)x^3 + 2C_3'(x) = 2x^4 \tag{5}$$

Multiply both sides of equation (4) by x and then subtract the respective sides of this equation from those of equation (3). Doing so eliminates $C_1'(x)$.

$$-C_2'(x)x^2 + 2C_3'(x)x^{-1} = 0$$

Solve this equation for $C_3'(x)$.

$$C_3'(x) = \frac{1}{2}C_2'(x)x^3$$

and then plug it into equation (5).

$$2C_2'(x)x^3 + 2\left[\frac{1}{2}C_2'(x)x^3\right] = 2x^4$$

$$3C_2'(x)x^3 = 2x^4$$

$$C_2'(x) = \frac{2}{3}x$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_2(x) = \frac{1}{3}x^2$$

Substitute this result into equation (5) to get $C_3(x)$.

$$2C_2'(x)x^3 + 2C_3'(x) = 2x^4 \quad \rightarrow \quad 2\left(\frac{2}{3}x\right)x^3 + 2C_3'(x) = 2x^4 \quad \rightarrow \quad \frac{4}{3}x^4 + 2C_3'(x) = 2x^4$$

$$C_3'(x) = \frac{1}{3}x^4$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_3(x) = \frac{1}{15}x^5$$

Substitute this result and the one for $C_2(x)$ into equation (3) to get $C_1(x)$.

$$C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^{-1} = 0 \quad \rightarrow \quad C_1'(x)x + \left(\frac{2}{3}x\right)x^2 + \left(\frac{1}{3}x^4\right)x^{-1} = 0$$

$$C_1'(x)x + \frac{2}{3}x^3 + \frac{1}{3}x^3 = 0$$

$$C_1'(x)x = -x^3$$

$$C_1'(x) = -x^2$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_1(x) = -\frac{1}{3}x^3$$

The particular solution is then

$$\begin{aligned} y_p(x) &= C_1(x)x + C_2(x)x^2 + C_3(x)x^{-1} \\ &= \left(-\frac{1}{3}x^3\right)x + \left(\frac{1}{3}x^2\right)x^2 + \left(\frac{1}{15}x^5\right)x^{-1} \\ &= -\frac{1}{3}x^4 + \frac{1}{3}x^4 + \frac{1}{15}x^4 \\ &= \frac{1}{15}x^4. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1x + C_2x^2 + C_3x^{-1} + \frac{1}{15}x^4. \end{aligned}$$