Problem 16

Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - 3y_c'' + 3y_c' - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt} \quad \rightarrow \quad y''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - 3(r^2e^{rt}) + 3(re^{rt}) - (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

$$r = \{1\}$$

One solution to equation (1) is then $y_c = e^t$. The $r = 1$ root has a multiplicity of 3, so a second and third linearly independent solution can be obtained from the first by including factors of t and t^2: $y_c = te^t$ and $y_c = t^2e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^t + C_2te^t + C_3t^2e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - 3y_p'' + 3y_p' - y_p = g(t). \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t$$

Substitute this into equation (2).

$$[C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t]''' - 3[C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t]''$$
$$+ 3[C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t]' - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t)$$

Evaluate the derivatives.

$$\begin{aligned}[C_1'(t)e^t + C_1(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t]'' \\ - 3[C_1'(t)e^t + C_1(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t]' \\ + 3[C_1'(t)e^t + C_1(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t)\end{aligned}$$

If we set $C_1'(t)e^t + C_2'(t)te^t + C_3'(t)t^2e^t = 0$, then this equation simplifies to

$$\begin{aligned}[C_1(t)e^t+C_2(t)e^t+C_2(t)te^t+2C_3(t)te^t+C_3(t)t^2e^t]''-3[C_1(t)e^t+C_2(t)e^t+C_2(t)te^t+2C_3(t)te^t+C_3(t)t^2e^t]' \\ + 3[C_1(t)e^t + C_2(t)e^t + C_2(t)te^t + 2C_3(t)te^t + C_3(t)t^2e^t] - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t)\end{aligned}$$

$$\begin{aligned}[C_1'(t)e^t + C_1(t)e^t + C_2'(t)e^t + C_2(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + 2C_3'(t)te^t \\ + 2C_3(t)e^t + 2C_3(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t]' \\ - 3[C_1'(t)e^t + C_1(t)e^t + C_2'(t)e^t + C_2(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + 2C_3'(t)te^t \\ + 2C_3(t)e^t + 2C_3(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ + 3[C_1(t)e^t + C_2(t)e^t + C_2(t)te^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t).\end{aligned}$$

If we set $C_1'(t)e^t + C_2'(t)e^t + C_2'(t)te^t + 2C_3'(t)te^t + C_3'(t)t^2e^t = 0$, then this equation simplifies to

$$\begin{aligned}[C_1(t)e^t + 2C_2(t)e^t + C_2(t)te^t + 2C_3(t)e^t + 4C_3(t)te^t + C_3(t)t^2e^t]' \\ - 3[C_1(t)e^t + 2C_2(t)e^t + C_2(t)te^t + 2C_3(t)e^t + 4C_3(t)te^t + C_3(t)t^2e^t] \\ + 3[C_1(t)e^t + C_2(t)e^t + C_2(t)te^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t)\end{aligned}$$

$$\begin{aligned}[C_1'(t)e^t + C_1(t)e^t + 2C_2'(t)e^t + 2C_2(t)e^t + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t + 2C_3'(t)e^t \\ + 2C_3(t)e^t + 4C_3'(t)te^t + 4C_3(t)e^t + 4C_3(t)te^t + C_3'(t)t^2e^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ - 3[C_1(t)e^t + 2C_2(t)e^t + C_2(t)te^t + 2C_3(t)e^t + 4C_3(t)te^t + C_3(t)t^2e^t] \\ + 3[C_1(t)e^t + C_2(t)e^t + C_2(t)te^t + 2C_3(t)te^t + C_3(t)t^2e^t] \\ - [C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t] = g(t)\end{aligned}$$

$$C_1'(t)e^t + C_2'(t)(2e^t + te^t) + C_3'(t)(2e^t + 4te^t + t^2e^t) = g(t).$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^t + C_2'(t)te^t + C_3'(t)t^2e^t = 0 \tag{3}$$

$$C_1'(t)e^t + C_2'(t)(e^t + te^t) + C_3'(t)(2te^t + t^2e^t) = 0 \tag{4}$$

$$C_1'(t)e^t + C_2'(t)(2e^t + te^t) + C_3'(t)(2e^t + 4te^t + t^2e^t) = g(t) \tag{5}$$

Subtract the respective sides of equation (3) from those of equation (4). Also, subtract the respective sides of equation (3) from those of equation (5). Doing so eliminates $C_1'(t)$.

$$\begin{gathered}C_2'(t)e^t + C_3'(t)(2te^t) = 0 \\ C_2'(t)(2e^t) + C_3'(t)(2e^t + 4te^t) = g(t)\end{gathered}$$

Solve this first equation for $C_2'(t)$

$$C_2'(t) = -2tC_3'(t) \tag{6}$$

and then plug it into the second equation.

$$\begin{gathered}[-2tC_3'(t)](2e^t) + C_3'(t)(2e^t + 4te^t) = g(t) \\ C_3'(t)(\cancel{-4te^t} + 2e^t + \cancel{4te^t}) = g(t) \\ C_3'(t) = \frac{1}{2}e^{-t}g(t)\end{gathered}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_3(t) = \int^t \frac{1}{2}e^{-s}g(s)\,ds$$

Substitute this result into equation (6) to get $C_2(t)$.

$$\begin{aligned}C_2'(t) &= -2t\left[\frac{1}{2}e^{-t}g(t)\right] \\ &= -te^{-t}g(t)\end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2(t) = \int^t -se^{-s}g(s)\,ds$$

Substitute this result and the one for $C_3(t)$ into equation (3) to get $C_1(t)$.

$$C_1'(t)e^t + C_2'(t)te^t + C_3'(t)t^2e^t = 0 \quad \rightarrow \quad C_1'(t)e^t + \left[-te^{-t}g(t)\right]te^t + \left[\frac{1}{2}e^{-t}g(t)\right]t^2e^t = 0$$

$$\begin{gathered}C_1'(t)e^t - t^2g(t) + \frac{1}{2}t^2g(t) = 0 \\ C_1'(t) = \frac{1}{2}t^2e^{-t}g(t)\end{gathered}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1(t) = \int^t \frac{1}{2}s^2e^{-s}g(s)\,ds$$

The particular solution is then

$$\begin{aligned}
y_p(t) &= C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t \\
&= \left[\int^t \frac{1}{2}s^2e^{-s}g(s)\,ds\right]e^t + \left[\int^t -se^{-s}g(s)\,ds\right]te^t + \left[\int^t \frac{1}{2}e^{-s}g(s)\,ds\right]t^2e^t \\
&= \frac{1}{2}\int^t s^2e^{t-s}g(s)\,ds - \int^t tse^{t-s}g(s)\,ds + \frac{1}{2}\int^t t^2e^{t-s}g(s)\,ds \\
&= \frac{1}{2}\int^t (s^2 - 2ts + t^2)e^{t-s}g(s)\,ds \\
&= \frac{1}{2}\int^t (s-t)^2e^{t-s}g(s)\,ds \\
&= \frac{1}{2}\int^t (t-s)^2e^{t-s}g(s)\,ds,
\end{aligned}$$

and the general solution is

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{1}{2}\int^t (t-s)^2e^{t-s}g(s)\,ds.
\end{aligned}$$

Suppose that $g(t) = t^{-2}e^t$. Then

$$\begin{aligned}
y(t) &= C_1e^t + C_2te^t + C_3t^2e^t + \frac{1}{2}\int^t (t-s)^2e^{t-s}s^{-2}e^s\,ds \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{1}{2}\int^t (t^2 - 2ts + s^2)e^ts^{-2}\,ds \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{e^t}{2}\int^t (t^2s^{-2} - 2ts^{-1} + 1)\,ds \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{e^t}{2}\left(-t^2s^{-1}\Big|^t - 2t\ln|s|\Big|^t + s\Big|^t\right) \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{e^t}{2}(-t - 2t\ln|t| + t) \\
&= C_1e^t + C_2te^t + C_3t^2e^t + \frac{e^t}{2}(-2t\ln|t|) \\
&= C_1e^t + C_2te^t + C_3t^2e^t - te^t\ln|t|.
\end{aligned}$$