Problem 17

Find a formula involving integrals for a particular solution of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = g(x), \qquad x > 0.$$

Hint: Verify that x, x^2, and x^3 are solutions of the homogeneous equation.

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(x)$ and $y_p(x)$, the complementary solution and the particular solution, respectively.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$x^3y_c''' - 3x^2y_c'' + 6xy_c' - 6y_c = 0 \tag{1}$$

Since this is an equidimensional equation, the solution is of the form $y_c = x^r$.

$$y_c = x^r \quad \rightarrow \quad y_c' = rx^{r-1} \quad \rightarrow \quad y_c'' = r(r-1)x^{r-2} \quad \rightarrow \quad y_c''' = r(r-1)(r-2)x^{r-3}$$

Substitute these expressions into the ODE.

$$x^3r(r-1)(r-2)x^{r-3} - 3x^2r(r-1)x^{r-2} + 6xrx^{r-1} - 6x^r = 0$$

$$r(r-1)(r-2)x^r - 3r(r-1)x^r + 6rx^r - 6x^r = 0$$

Divide both sides by x^r.

$$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$(r-1)(r-2)(r-3) = 0$$

$$r = \{1, 2, 3\}$$

Three solutions to equation (1) are then $y_c = x^1 = x$ and $y_c = x^2$ and $y_c = x^3$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(x) = C_1x + C_2x^2 + C_3x^3$$

On the other hand, the particular solution satisfies

$$x^3y_p''' - 3x^2y_p'' + 6xy_p' - 6y_p = g(x). \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(x)$ to vary.

$$y_p(x) = C_1(x)x + C_2(x)x^2 + C_3(x)x^3$$

Substitute this into equation (2).

$$x^3[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]''' - 3x^2[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]'' \\ + 6x[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]' - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x)$$

Evaluate the derivatives.

$$\begin{aligned} x^3[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2]'' \\ - 3x^2[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2]' \\ + 6x[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2] \\ - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \end{aligned}$$

If we set $C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^3 = 0$, then this equation simplifies to

$$\begin{aligned} x^3[C_1(x) + 2C_2(x)x + 3C_3(x)x^2]'' - 3x^2[C_1(x) + 2C_2(x)x + 3C_3(x)x^2]' \\ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \end{aligned}$$

$$\begin{aligned} x^3[C_1'(x) + 2C_2'(x)x + 2C_2(x) + 3C_3'(x)x^2 + 6C_3(x)x]' \\ - 3x^2[C_1'(x) + 2C_2'(x)x + 2C_2(x) + 3C_3'(x)x^2 + 6C_3(x)x] \\ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] \\ - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x). \end{aligned}$$

If we set $C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 = 0$, then this equation simplifies to

$$\begin{aligned} x^3[2C_2(x) + 6C_3(x)x]' - 3x^2[2C_2(x) + 6C_3(x)x] \\ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \end{aligned}$$

$$\begin{aligned} x^3[2C_2'(x) + 6C_3'(x)x + 6C_3(x)] - 3x^2[2C_2(x) + 6C_3(x)x] \\ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \end{aligned}$$

$$2C_2'(x)x^3 + 6C_3'(x)x^4 = g(x).$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^3 = 0 \tag{3}$$

$$C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 = 0 \tag{4}$$

$$2C_2'(x)x^3 + 6C_3'(x)x^4 = g(x) \tag{5}$$

Multiply both sides of equation (4) by x and then subtract the respective sides of it from equation (3) to eliminate $C_1'(x)$.

$$-C_2'(x)x^2 - 2C_3'(x)x^3 = 0$$

Solve this equation for $C_2'(x)$

$$C_2'(x) = -2C_3'(x)x \tag{6}$$

and then plug it into equation (5).

$$2[-2C_3'(x)x]x^3 + 6C_3'(x)x^4 = g(x)$$

$$2C_3'(x)x^4 = g(x)$$

$$C_3'(x) = \frac{1}{2}x^{-4}g(x)$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_3(x) = \int^x \frac{1}{2}s^{-4}g(s)\,ds$$

Substitute this result into equation (6) to get $C_2(x)$.

$$\begin{aligned} C_2'(x) &= -2\left[\frac{1}{2}x^{-4}g(x)\right]x \\ &= -x^{-3}g(x) \end{aligned}$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_2(x) = \int^x -s^{-3}g(s)\,ds$$

Substitute this result and the one for $C_3(x)$ into equation (4) to get $C_1(x)$.

$$C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 = 0 \quad \rightarrow \quad C_1'(x) + 2[-x^{-3}g(x)]x + 3\left[\frac{1}{2}x^{-4}g(x)\right]x^2 = 0$$

$$C_1'(x) - 2x^{-2}g(x) + \frac{3}{2}x^{-2}g(x) = 0$$

$$C_1'(x) = \frac{1}{2}x^{-2}g(x)$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_1(x) = \int^x \frac{1}{2}s^{-2}g(s)\,ds$$

The particular solution is then

$$\begin{aligned} y_p(x) &= C_1(x)x + C_2(x)x^2 + C_3(x)x^3 \\ &= \left[\int^x \frac{1}{2}s^{-2}g(s)\,ds\right]x + \left[\int^x -s^{-3}g(s)\,ds\right]x^2 + \left[\int^x \frac{1}{2}s^{-4}g(s)\,ds\right]x^3 \\ &= \frac{1}{2}\int^x xs^{-2}g(s)\,ds - \int^x x^2s^{-3}g(s)\,ds + \frac{1}{2}\int^x x^3s^{-4}g(s)\,ds \\ &= \frac{1}{2}\int^x \left(\frac{x}{s^2} - \frac{2x^2}{s^3} + \frac{x^3}{s^4}\right)g(s)\,ds \\ &= \frac{1}{2}\int^x \left(\frac{xs^2 - 2x^2s + x^3}{s^4}\right)g(s)\,ds \\ &= \frac{1}{2}\int^x x\left(\frac{s^2 - 2xs + x^2}{s^4}\right)g(s)\,ds \\ &= \frac{1}{2}\int^x x\frac{(s-x)^2}{s^4}g(s)\,ds \\ &= \frac{x}{2}\int^x \frac{(x-s)^2}{s^4}g(s)\,ds, \end{aligned}$$

and the general solution is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1x + C_2x^2 + C_3x^3 + \frac{x}{2}\int^x \frac{(x-s)^2}{s^4}g(s)\,ds. \end{aligned}$$