Problem 3

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' - 2y'' - y' + 2y = e^{4t}$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - 2y_c'' - y_c' + 2y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - 2(r^2e^{rt}) - re^{rt} + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 2r^2 - r + 2 = 0$$

$$(r-2)(r^2-1) = 0$$

$$r = \{-1, 1, 2\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^t$ and $y_c = e^{2t}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1e^{-t} + C_2e^t + C_3e^{2t}$$

On the other hand, the particular solution satisfies

$$y_p''' - 2y_p'' - y_p' + 2y_p = e^{4t}. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}$$

Substitute this formula into equation (2).

$$[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]''' - 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]''$$
$$- [C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]' + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}$$

Evaluate the derivatives.

$$\begin{aligned}[]
[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}]'' \\
- 2[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}]' \\
- [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}] \\
+ 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}
\end{aligned}$$

If we set $C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0$, then this equation simplifies to

$$\begin{aligned}[]
[-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}]'' - 2[-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}]' \\
- [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}
\end{aligned}$$

$$\begin{aligned}[]
[-C_1'(t)e^{-t} + C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + 2C_3'(t)e^{2t} + 4C_3(t)e^{2t}]' \\
- 2[-C_1'(t)e^{-t} + C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + 2C_3'(t)e^{2t} + 4C_3(t)e^{2t}] \\
- [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] \\
+ 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}.
\end{aligned}$$

If we set $-C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t} = 0$, then this equation simplifies to

$$\begin{aligned}[]
[C_1(t)e^{-t} + C_2(t)e^t + 4C_3(t)e^{2t}]' - 2[C_1(t)e^{-t} + C_2(t)e^t + 4C_3(t)e^{2t}] \\
- [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}
\end{aligned}$$

$$\begin{aligned}[]
[C_1'(t)e^{-t} - \cancel{C_1(t)e^{-t}} + C_2'(t)e^t + \cancel{C_2(t)e^t} + 4C_3'(t)e^{2t} + \cancel{8C_3(t)e^{2t}}] - 2[\cancel{C_1(t)e^{-t}} + \cancel{C_2(t)e^t} + \cancel{4C_3(t)e^{2t}}] \\
- [-\cancel{C_1(t)e^{-t}} + \cancel{C_2(t)e^t} + \cancel{2C_3(t)e^{2t}}] + 2[\cancel{C_1(t)e^{-t}} + \cancel{C_2(t)e^t} + \cancel{C_3(t)e^{2t}}] = e^{4t}
\end{aligned}$$

$$C_1'(t)e^{-t} + C_2'(t)e^t + 4C_3'(t)e^{2t} = e^{4t}.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0 \tag{3}$$
$$-C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t} = 0 \tag{4}$$
$$C_1'(t)e^{-t} + C_2'(t)e^t + 4C_3'(t)e^{2t} = e^{4t} \tag{5}$$

Multiply both sides of equation (3) by 2 and add the respective sides to equation (4) and subtract the respective sides from those of equation (5).

$$2[\cancel{C_1'(t)e^{-t}} + C_2'(t)e^t + \cancel{C_3'(t)e^{2t}}] + [-\cancel{C_1'(t)e^{-t}} + C_2'(t)e^t + \cancel{2C_3'(t)e^{2t}}] - [\cancel{C_1'(t)e^{-t}} + C_2'(t)e^t + \cancel{4C_3'(t)e^{2t}}] = -e^{4t}$$

$$2C_2'(t)e^t = -e^{4t}$$

$$C_2'(t) = -\frac{1}{2}e^{3t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2(t) = -\frac{1}{6}e^{3t}$$

Substitute this result back into equations (3) and (4).

$$C_1'(t)e^{-t} + \left[-\frac{1}{2}e^{3t}\right]e^t + C_3'(t)e^{2t} = 0$$
$$-C_1'(t)e^{-t} + \left[-\frac{1}{2}e^{3t}\right]e^t + 2C_3'(t)e^{2t} = 0$$

Bring each second term to the right side and then add the respective sides of these equations.

$$3C_3'(t)e^{2t} = (e^{3t})e^t$$

$$C_3'(t) = \frac{1}{3}e^{2t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_3(t) = \frac{1}{6}e^{2t}$$

Now substitute this result and the one for $C_4(t)$ into equation (3) to solve for $C_1(t)$.

$$C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0 \quad \rightarrow \quad C_1'(t)e^{-t} + \left(-\frac{1}{2}e^{3t}\right)e^t + \left(\frac{1}{3}e^{2t}\right)e^{2t} = 0$$

$$C_1'(t)e^{-t} - \frac{1}{2}e^{4t} + \frac{1}{3}e^{4t} = 0$$
$$C_1'(t)e^{-t} = \frac{1}{6}e^{4t}$$
$$C_1'(t) = \frac{1}{6}e^{5t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1(t) = \frac{1}{30}e^{5t}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t} \\ &= \left(\frac{1}{30}e^{5t}\right)e^{-t} + \left(-\frac{1}{6}e^{3t}\right)e^t + \left(\frac{1}{6}e^{2t}\right)e^{2t} \\ &= \frac{1}{30}e^{4t} - \cancel{\frac{1}{6}e^{4t}} + \cancel{\frac{1}{6}e^{4t}} \\ &= \frac{1}{30}e^{4t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{-t} + C_2e^t + C_3e^{2t} + \frac{1}{30}e^{4t}. \end{aligned}$$