# Problem 6

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

$$y^{(4)} + 2y'' + y = \sin t$$

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**Solution**

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c^{(4)} + 2y_c'' + y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt} \quad \rightarrow \quad y_c^{(4)} = r^4e^{rt}$$

Substitute these expressions into the ODE.

$$r^4e^{rt} + 2(r^2e^{rt}) + e^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^4 + 2r^2 + 1 = 0$$
$$(r^2 + 1)^2 = 0$$
$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. Since the multiplicity of each root is 2, a second linearly independent solution can be obtained from each one by including a factor of $t$: $y_c = te^{-it}$ and $y_c = te^{it}$. By the principle of superposition, the general solution for $y_c$ is a linear combination of these four.

$$\begin{aligned} y_c(t) &= C_1e^{-it} + C_2e^{it} + C_3te^{-it} + C_4te^{it} \\ &= C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t) + C_3t(\cos t - i\sin t) + C_4t(\cos t + i\sin t) \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t + t(C_3 + C_4)\cos t + t(-iC_3 + iC_4)\sin t \\ &= C_5\cos t + C_6\sin t + C_7t\cos t + C_8t\sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p^{(4)} + 2y_p'' + y_p = \sin t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t$$

Substitute this formula into equation (2).

$$\begin{aligned}&[C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t]^{(4)}\\ &\qquad + 2[C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t]''\\ &\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t\end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}&[C_5'(t)\cos t - C_5(t)\sin t + C_6'(t)\sin t + C_6(t)\cos t + C_7'(t)t\cos t + C_7(t)\cos t - C_7(t)t\sin t\\ &+ C_8'(t)t\sin t + C_8(t)\sin t + C_8(t)t\cos t]''' + 2[C_5'(t)\cos t - C_5(t)\sin t + C_6'(t)\sin t + C_6(t)\cos t\\ &\qquad + C_7'(t)t\cos t + C_7(t)\cos t - C_7(t)t\sin t + C_8'(t)t\sin t + C_8(t)\sin t + C_8(t)t\cos t]'\\ &\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t\end{aligned}$$

If we set $C_5'(t)\cos t + C_6'(t)\sin t + C_7'(t)t\cos t + C_8'(t)t\sin t = 0$, then this equation simplifies to

$$\begin{aligned}&[-C_5(t)\sin t + C_6(t)\cos t + C_7(t)\cos t - C_7(t)t\sin t + C_8(t)\sin t + C_8(t)t\cos t]'''\\ &\qquad + 2[-C_5(t)\sin t + C_6(t)\cos t + C_7(t)\cos t - C_7(t)t\sin t + C_8(t)\sin t + C_8(t)t\cos t]'\\ &\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t\end{aligned}$$

$$\begin{aligned}&[-C_5'(t)\sin t - C_5(t)\cos t + C_6'(t)\cos t - C_6(t)\sin t + C_7'(t)\cos t - C_7(t)\sin t - C_7'(t)t\sin t - C_7(t)\sin t\\ &\qquad - C_7(t)t\cos t + C_8'(t)\sin t + C_8(t)\cos t + C_8'(t)t\cos t + C_8(t)\cos t - C_8(t)t\sin t]''\\ &+2[-C_5'(t)\sin t - C_5(t)\cos t + C_6'(t)\cos t - C_6(t)\sin t + C_7'(t)\cos t - C_7(t)\sin t - C_7'(t)t\sin t - C_7(t)\sin t\\ &\qquad - C_7(t)t\cos t + C_8'(t)\sin t + C_8(t)\cos t + C_8'(t)t\cos t + C_8(t)\cos t - C_8(t)t\sin t]\\ &\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t.\end{aligned}$$

If we set $-C_5'(t)\sin t + C_6'(t)\cos t + C_7'(t)\cos t - C_7'(t)t\sin t + C_8'(t)\sin t + C_8'(t)t\cos t = 0$, then this equation simplifies to

$$\begin{aligned}&[-C_5(t)\cos t - C_6(t)\sin t - 2C_7(t)\sin t\\ &\qquad\qquad - C_7(t)t\cos t + 2C_8(t)\cos t - C_8(t)t\sin t]''\\ &\qquad + 2[-C_5(t)\cos t - C_6(t)\sin t - 2C_7(t)\sin t\\ &\qquad\qquad - C_7(t)t\cos t + 2C_8(t)\cos t - C_8(t)t\sin t]\\ &\qquad\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t\end{aligned}$$

$$\begin{aligned}&[-C_5'(t)\cos t + C_5(t)\sin t - C_6'(t)\sin t - C_6(t)\cos t - 2C_7'(t)\sin t - 2C_7(t)\cos t - C_7'(t)t\cos t\\ &- C_7(t)\cos t + C_7(t)t\sin t + 2C_8'(t)\cos t - 2C_8(t)\sin t - C_8'(t)t\sin t - C_8(t)\sin t - C_8(t)t\cos t]'\\ &\qquad + 2[-C_5(t)\cos t - C_6(t)\sin t - 2C_7(t)\sin t\\ &\qquad\qquad - C_7(t)t\cos t + 2C_8(t)\cos t - C_8(t)t\sin t]\\ &\qquad\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t.\end{aligned}$$

If we set $-C_5'(t)\cos t - C_6'(t)\sin t - 2C_7'(t)\sin t - C_7'(t)t\cos t + 2C_8'(t)\cos t - C_8'(t)t\sin t = 0$, then this equation simplifies to

$$\begin{aligned}&[C_5(t)\sin t - C_6(t)\cos t - 2C_7(t)\cos t - C_7(t)\cos t + C_7(t)t\sin t - 2C_8(t)\sin t - C_8(t)\sin t - C_8(t)t\cos t]'\\ &\qquad + 2[-C_5(t)\cos t - C_6(t)\sin t - 2C_7(t)\sin t\\ &\qquad\qquad - C_7(t)t\cos t + 2C_8(t)\cos t - C_8(t)t\sin t]\\ &\qquad\qquad\qquad + [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t\end{aligned}$$

$$\begin{aligned}
[C_5'(t)\sin t + C_5(t)\cos t - C_6'(t)\cos t + C_6(t)\sin t - 2C_7'(t)\cos t + 2C_7(t)\sin t - C_7'(t)\cos t + C_7(t)\sin t \\
+ C_7'(t)t\sin t + C_7(t)\sin t + C_7(t)t\cos t - 2C_8'(t)\sin t - 2C_8(t)\cos t - C_8'(t)\sin t \\
- C_8(t)\cos t - C_8'(t)t\cos t - C_8(t)\cos t + C_8(t)t\sin t] \\
+ 2[-C_5(t)\cos t - C_6(t)\sin t - 2C_7(t)\sin t \\
- C_7(t)t\cos t + 2C_8(t)\cos t - C_8(t)t\sin t] \\
+ [C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t] = \sin t
\end{aligned}$$

$$C_5'(t)\sin t - C_6'(t)\cos t + C_7'(t)(t\sin t - 3\cos t) + C_8'(t)(-3\sin t - t\cos t) = \sin t$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_5'(t)\cos t + C_6'(t)\sin t + C_7'(t)t\cos t + C_8'(t)t\sin t = 0 \tag{3}$$
$$-C_5'(t)\sin t + C_6'(t)\cos t + C_7'(t)(\cos t - t\sin t) + C_8'(t)(\sin t + t\cos t) = 0 \tag{4}$$
$$-C_5'(t)\cos t - C_6'(t)\sin t + C_7'(t)(-2\sin t - t\cos t) + C_8'(t)(2\cos t - t\sin t) = 0 \tag{5}$$
$$C_5'(t)\sin t - C_6'(t)\cos t + C_7'(t)(t\sin t - 3\cos t) + C_8'(t)(-3\sin t - t\cos t) = \sin t \tag{6}$$

Add the respective sides of equations (3) and (5) together, and add the respective sides of equations (4) and (6) together. Doing so eliminates $C_5'(t)$ and $C_6'(t)$.

$$\begin{aligned}
C_7'(t)(-2\sin t) + C_8'(t)(2\cos t) = 0 \\
C_7'(t)(-2\cos t) + C_8'(t)(-2\sin t) = \sin t
\end{aligned}$$

Solve this first equation for $C_8'(t)$

$$C_8'(t) = \frac{\sin t}{\cos t}C_7'(t) \tag{7}$$

and plug it into the second equation.

$$C_7'(t)(-2\cos t) + \left[\frac{\sin t}{\cos t}C_7'(t)\right](-2\sin t) = \sin t$$

$$C_7'(t)\left(\cos t + \frac{\sin t}{\cos t}\sin t\right) = -\frac{1}{2}\sin t$$

$$C_7'(t)\left(\frac{\cos^2 t + \sin^2 t}{\cos t}\right) = -\frac{1}{2}\sin t$$

$$C_7'(t)\left(\frac{1}{\cos t}\right) = -\frac{1}{2}\sin t$$

$$\begin{aligned}
C_7'(t) &= -\frac{1}{2}\sin t\cos t \\
&= -\frac{1}{4}\sin 2t
\end{aligned}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_7(t) = \frac{1}{8}\cos 2t$$

Substitute this back into equation (7) to get $C_8(t)$.

$$\begin{aligned} C_8'(t) &= \frac{\sin t}{\cos t}\left(-\frac{1}{2}\sin t\cos t\right) \\ &= -\frac{1}{2}\sin^2 t \\ &= -\frac{1}{4}(1-\cos 2t) \end{aligned}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_8(t) = \frac{1}{8}(\sin 2t - 2t)$$

Substitute the results for $C_7'(t)$ and $C_8'(t)$ into equations (3) and (4).

$$\begin{aligned} C_5'(t)\cos t + C_6'(t)\sin t + \left(-\frac{1}{2}\sin t\cos t\right)t\cos t + \left(-\frac{1}{2}\sin^2 t\right)t\sin t &= 0 \\ -C_5'(t)\sin t + C_6'(t)\cos t + \left(-\frac{1}{2}\sin t\cos t\right)(\cos t - t\sin t) + \left(-\frac{1}{2}\sin^2 t\right)(\sin t + t\cos t) &= 0 \end{aligned}$$

Simplify these equations.

$$\begin{aligned} C_5'(t)\cos t + C_6'(t)\sin t &= \frac{1}{2}t\sin t \qquad (8) \\ -C_5'(t)\sin t + C_6'(t)\cos t &= \frac{1}{2}\sin t \end{aligned}$$

Solve this second equation for $C_6'(t)$

$$C_6'(t) = \frac{1}{\cos t}\left[\frac{1}{2}\sin t + C_5'(t)\sin t\right]$$

and plug it into the first equation.

$$\begin{aligned} C_5'(t)\cos t + \frac{1}{\cos t}\left[\frac{1}{2}\sin t + C_5'(t)\sin t\right]\sin t &= \frac{1}{2}t\sin t \\ C_5'(t)\left(\cos t + \frac{\sin^2 t}{\cos t}\right) + \frac{1}{2}\frac{\sin^2 t}{\cos t} &= \frac{1}{2}t\sin t \\ C_5'(t)\left(\frac{\cos^2 t + \sin^2 t}{\cos t}\right) &= \frac{1}{2}t\sin t - \frac{1}{2}\frac{\sin^2 t}{\cos t} \\ C_5'(t)\left(\frac{1}{\cos t}\right) &= \frac{1}{2}t\sin t - \frac{1}{2}\frac{\sin^2 t}{\cos t} \end{aligned}$$

$$\begin{aligned} C_5'(t) &= \frac{1}{2}t\sin t\cos t - \frac{1}{2}\sin^2 t \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}(1-\cos 2t) \end{aligned}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$\begin{aligned} C_5(t) &= \frac{1}{16}(\sin 2t - 2t\cos 2t) - \frac{t}{4} + \frac{1}{8}\sin 2t \\ &= \frac{1}{16}(3\sin 2t - 2t\cos 2t) - \frac{t}{4} \end{aligned}$$

Substitute this result into equation (8) to determine $C_6(t)$.

$$\left(\frac{1}{2}t\sin t\cos t - \frac{1}{2}\sin^2 t\right)\cos t + C_6'(t)\sin t = \frac{1}{2}t\sin t$$

$$\left(\frac{1}{2}t\cos t - \frac{1}{2}\sin t\right)\cos t + C_6'(t) = \frac{1}{2}t$$

$$\frac{1}{2}t\cos^2 t - \frac{1}{2}\sin t\cos t + C_6'(t) = \frac{1}{2}t$$

$$\begin{aligned} C_6'(t) &= \frac{1}{2}t(1-\cos^2 t) + \frac{1}{2}\sin t\cos t \\ &= \frac{1}{2}t\sin^2 t + \frac{1}{4}\sin 2t \\ &= \frac{1}{4}t(1-\cos 2t) + \frac{1}{4}\sin 2t \\ &= \frac{t}{4} - \frac{t}{4}\cos 2t + \frac{1}{4}\sin 2t \end{aligned}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$\begin{aligned} C_6(t) &= \frac{t^2}{8} - \frac{1}{16}(2t\sin 2t + \cos 2t) - \frac{1}{8}\cos 2t \\ &= \frac{t^2}{8} - \frac{1}{16}(2t\sin 2t + 3\cos 2t) \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_5(t)\cos t + C_6(t)\sin t + C_7(t)t\cos t + C_8(t)t\sin t \\ &= \left[\frac{1}{16}(3\sin 2t - 2t\cos 2t) - \frac{t}{4}\right]\cos t + \left[\frac{t^2}{8} - \frac{1}{16}(2t\sin 2t + 3\cos 2t)\right]\sin t \\ &\qquad + \left(\frac{1}{8}\cos 2t\right)t\cos t + \left[\frac{1}{8}(\sin 2t - 2t)\right]t\sin t \\ &= -\frac{t}{4}\cos t + \frac{3}{16}\sin t - \frac{t^2}{8}\sin t. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_5\cos t + C_6\sin t + C_7 t\cos t + C_8 t\sin t - \frac{t}{4}\cos t + \frac{3}{16}\sin t - \frac{t^2}{8}\sin t \\ &= C_5\cos t + \left(C_6 + \frac{3}{16}\right)\sin t + \left(C_7 - \frac{1}{4}\right)t\cos t + C_8 t\sin t - \frac{t^2}{8}\sin t \\ &= C_5\cos t + C_9\sin t + C_{10}t\cos t + C_8 t\sin t - \frac{t^2}{8}\sin t. \end{aligned}$$