

Problem 8

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

$$y''' - y' = \csc t, \quad 0 < t < \pi$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c' = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \rightarrow y_c' = r e^{rt} \rightarrow y_c'' = r^2 e^{rt} \rightarrow y_c''' = r^3 e^{rt}$$

Substitute these expressions into the ODE.

$$r^3 e^{rt} - r e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$

$$r = \{-1, 0, 1\}$$

Three solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^0 = 1$ and $y_c = e^t$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$y_c(t) = C_1 e^{-t} + C_2 + C_3 e^t$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p' = \csc t. \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{-t} + C_2(t) + C_3(t)e^t$$

Substitute this formula into equation (2).

$$[C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]''' - [C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]' = \csc t$$

Evaluate the derivatives.

$$[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t]'' - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t + C_3(t)e^t]' = \csc t$$

If we set $C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0$, then this equation simplifies to

$$[-C_1(t)e^{-t} + C_3(t)e^t]'' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t$$

$$[-C_1'(t)e^{-t} + C_1(t)e^{-t} + C_3'(t)e^t + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t.$$

If we set $-C_1'(t)e^{-t} + C_3'(t)e^t = 0$, then this equation simplifies to

$$[C_1(t)e^{-t} + C_3(t)e^t]' - [-C_1(t)e^{-t} + C_3(t)e^t] = \csc t$$

$$[C_1'(t)e^{-t} - \cancel{C_1(t)e^{-t}} + C_3'(t)e^t + \cancel{C_3(t)e^t}] - [-\cancel{C_1(t)e^{-t}} + \cancel{C_3(t)e^t}] = \csc t$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = \csc t.$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \quad (3)$$

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \quad (4)$$

$$C_1'(t)e^{-t} + C_3'(t)e^t = \csc t \quad (5)$$

Start by solving equation (4) for $C_1'(t)$.

$$C_1'(t) = e^{2t}C_3'(t)$$

and then plugging it in to equation (5).

$$[e^{2t}C_3'(t)]e^{-t} + C_3'(t)e^t = \csc t$$

$$2C_3'(t)e^t = \csc t$$

$$C_3'(t) = \frac{1}{2}e^{-t} \csc t$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_3(t) = \int \frac{1}{2}e^{-s} \csc s \, ds$$

Substitute this back into equation (4) to get $C_1(t)$.

$$-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \left[\frac{1}{2}e^{-t} \csc t \right] e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \frac{1}{2} \csc t = 0$$

$$C_1'(t) = \frac{1}{2}e^t \csc t$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_1(t) = \int \frac{1}{2}e^s \csc s \, ds$$

Substitute this result along with $C_3(t)$ into equation (3) to obtain $C_2(t)$.

$$C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \quad \rightarrow \quad \left[\frac{1}{2}e^t \csc t \right] e^{-t} + C_2'(t) + \left[\frac{1}{2}e^{-t} \csc t \right] e^t = 0 \quad \rightarrow \quad \frac{1}{2} \csc t + C_2'(t) + \frac{1}{2} \csc t = 0$$

$$C_2'(t) = -\csc t$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_2(t) = \ln|\csc t + \cot t|$$

Because $0 < t < \pi$, the absolute value sign can be dropped. The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)e^{-t} + C_2(t) + C_3(t)e^t \\ &= \left(\int^t \frac{1}{2} e^s \csc s \, ds \right) e^{-t} + \ln(\csc t + \cot t) + \left(\int^t \frac{1}{2} e^{-s} \csc s \, ds \right) e^t \\ &= \frac{1}{2} \int^t e^{-(t-s)} \csc s \, ds + \ln(\csc t + \cot t) + \frac{1}{2} \int^t e^{t-s} \csc s \, ds \\ &= \frac{1}{2} \int^t \left[e^{-(t-s)} \csc s + e^{t-s} \csc s \right] ds + \ln(\csc t + \cot t) \\ &= \int^t \frac{e^{-(t-s)} + e^{t-s}}{2} \csc s \, ds + \ln(\csc t + \cot t) \\ &= \int^t \cosh(t-s) \csc s \, ds + \ln(\csc t + \cot t). \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 e^{-t} + C_2 + C_3 e^t + \int^t \cosh(t-s) \csc s \, ds + \ln(\csc t + \cot t). \end{aligned}$$