Problem 2

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$$

Solution

Apply the ratio test to determine the condition in which the provided series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}x^{n+1}}{\frac{n}{2^n}x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{2}\frac{n+1}{n}x\right| \\
&= \lim_{n\to\infty}\frac{1}{2}\frac{n+1}{n}|x| \\
&= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)|x| \\
&= \frac{1}{2}|x|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } \frac{1}{2}|x| < 1 \\ \text{unknown} & \text{if } \frac{1}{2}|x| = 1 \\ \text{divergent} & \text{if } \frac{1}{2}|x| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x| < 2$, or $-2 < x < 2$, we see that the center of convergence is at $x = 0$ and the radius of convergence is 2.