Problem 6

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

$$\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n}$$

Solution

Apply the ratio test to determine the condition in which the provided series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(x-x_0)^{n+1}}{n+1}}{\frac{(x-x_0)^n}{n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n}{n+1}(x-x_0)\right| \\
&= \lim_{n\to\infty}\frac{n}{n+1}|x-x_0| \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x-x_0| \\
&= |x-x_0|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |x-x_0|<1 \\ \text{unknown} & \text{if } |x-x_0|=1 \\ \text{divergent} & \text{if } |x-x_0|>1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < x - x_0 < 1$, or $-1 + x_0 < x < 1 + x_0$, we see that the center of convergence is at $x = x_0$ and the radius of convergence is 1.