Problem 7

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$$

Solution

Apply the ratio test to determine the condition in which the provided series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{3^{n+1}}}{\frac{(-1)^n n^2 (x+2)^n}{3^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\frac{(n+1)^2}{n^2}\frac{3^n}{3^{n+1}}\frac{(x+2)^{n+1}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)\frac{n^2\left(1+\frac{1}{n}\right)^2}{n^2}\frac{1}{3}(x+2)\right| \\
&= \lim_{n\to\infty}\left|(-1)\left(1+\frac{1}{n}\right)^2\frac{1}{3}(x+2)\right| \\
&= \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^2|x+2| \\
&= \frac{1}{3}|x+2|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } \frac{1}{3}|x+2| < 1 \\ \text{unknown} & \text{if } \frac{1}{3}|x+2| = 1 \\ \text{divergent} & \text{if } \frac{1}{3}|x+2| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x+2| < 3$, or $-3 < x+2 < 3$, we see that the center of convergence is at $x = -2$ and the radius of convergence is 3.